Binomial Theorem

If a and x are two real number and n is a positive integer then

$$(a+x)^n=\binom{n}{0}a^n+\binom{n}{1}a^{n-1}x^1+\binom{n}{2}a^{n-2}x^2+\ldots+\binom{n}{n-1}a\,x^{n-1}+\binom{n}{n}x^n$$

Proof

We will use mathematical induction to prove this so let $S(n)$ be the given statement.

Put $n=1$

$$S(1):(a+x)^1=\binom{1}{0}a^1+\binom{1}{1}a^{1-1}\,x^1=(1)\,a+(1)(1)\;x \quad\Rightarrow\; a+x=a+x$$

$S(1)$ is true so condition I is satisfied.

Now suppose that $S(n)$ is true for $n=k$.

$$S(k):\;(a+x)^k=\binom{k}{0}a^k+\binom{k}{1}a^{k-1}x^1+\binom{k}{2}a^{k-2}x^2+\ldots+\binom{k}{k-1}a\,x^{k-1}+\binom{k}{k}x^k \;\ldots..(i)$$

The statement for $n=k+1$

$$S(k+1):\;(a+x)^{k+1}=\binom{k+1}{0}a^{k+1}+\binom{k+1}{1}a^{k+1-1}x^1+\binom{k+1}{2}a^{k+1-2}x^2+\ldots$$

$$+\binom{k+1}{k+1-1}a\,x^{k+1-1}+\binom{k+1}{k+1}x^{k+1}$$

$$\Rightarrow\;(a+x)^{k+1}=\binom{k+1}{0}a^{k+1}+\binom{k+1}{1}a^kx^1+\binom{k+1}{2}a^{k-1}x^2+\ldots$$

$$+\binom{k+1}{k}a\,x^k+\binom{k+1}{k+1}x^{k+1}$$

Multiplying both sides of equation (i) by $(a+x)$

$$(a+x)^k(a+x)=\left(\binom{k}{0}a^k+\binom{k}{1}a^{k-1}x^1+\binom{k}{2}a^{k-2}x^2+\ldots+\binom{k}{k-1}a\,x^{k-1}+\binom{k}{k}x^k\right)(a+x)$$

$$\Rightarrow\;(a+x)^{k+1}=\left(\binom{k}{0}a^k+\binom{k}{1}a^{k-1}x^1+\binom{k}{2}a^{k-2}x^2+\ldots+\binom{k}{k-1}a\,x^{k-1}+\binom{k}{k}x^k\right)(a)$$

$$+\left(\binom{k}{0}a^k+\binom{k}{1}a^{k-1}x^1+\binom{k}{2}a^{k-2}x^2+\ldots+\binom{k}{k-1}a\,x^{k-1}+\binom{k}{k}x^k\right)(x)$$

$$\Rightarrow\;(a+x)^{k+1}=\binom{k}{0}a^{k+1}+\binom{k}{1}a^kx^1+\binom{k}{2}a^{k-1}x^2+\ldots+\binom{k}{k-1}a^2\,x^{k-1}+\binom{k}{k}a\,x^k$$

$$+\binom{k}{0}a^kx+\binom{k}{1}a^{k-1}x^2+\binom{k}{2}a^{k-2}x^3+\ldots+\binom{k}{k-1}a\,x^k+\binom{k}{k}x^{k+1}$$

$$\Rightarrow\;(a+x)^{k+1}=\binom{k}{0}a^{k+1}+\left(\binom{k}{1}+\binom{k}{0}\right)a^kx^1+\left(\binom{k}{2}+\binom{k}{1}\right)a^{k-1}x^2+\ldots$$

$$+\left(\binom{k}{k}+\binom{k}{k-1}\right)a\,x^k+\binom{k}{k}x^{k+1}$$

Since $\binom{n}{0}=\binom{n+1}{0}$, $\binom{n}{r}+\binom{n}{r-1}=\binom{n+1}{r}$ and $\binom{n}{n}=\binom{n+1}{n+1}$

$$\Rightarrow\ (a+x)^{k+1}=\binom{k+1}{0}a^{k+1}+\binom{k+1}{1}a^kx^1+\binom{k+1}{2}a^{k-1}x^2+\ldots+\binom{k+1}{k}a\,x^k+\binom{k+1}{k+1}x^{k+1}$$

Thus $S(k+1)$ is true when $S(k)$ is true so condition II is satisfied and $S(n)$ is true for all

positive integral value of *n*.

Question # 1

Using binomial theorem, expand the following:

(i) $(a+2b)^5$ (ii) $\left(\frac{x}{2}-\frac{2}{x^2}\right)^6$ (iii) $\left(3a-\frac{x}{3a}\right)^4$

(iv) $\left(2a-\frac{x}{a}\right)^7$ (v) $\left(\frac{x}{2y}-\frac{2y}{x}\right)^8$ (vi) $\left(\sqrt{\frac{a}{x}}-\sqrt{\frac{x}{a}}\right)^6$

Solution

(i)

$$(a+2b)^5=\binom{5}{0}a^5+\binom{5}{1}a^{5-1}(2b)^1+\binom{5}{2}a^{5-2}(2b)^2+\binom{5}{3}a^{5-3}(2b)^3+\binom{5}{4}a^{5-4}(2b)^4+\binom{5}{5}a^{5-5}(2b)^5$$

$$=(1)\,a^5+(5)\,a^4(2b)+(10)\,a^3(4b^2)+(10)\,a^2(8b^3)+(5)\,a^1(16b^4)+(1)\,a^0(32b^5)$$

$$=a^5+10a^4b+40a^3b^2+80a^2b^3+80ab^4+32b^5 \qquad \because\ a^0=1$$

(ii) $$\left(\frac{x}{2}-\frac{2}{x^2}\right)^6=\binom{6}{0}\left(\frac{x}{2}\right)^6+\binom{6}{1}\left(\frac{x}{2}\right)^{6-1}\left(-\frac{2}{x^2}\right)^1+\binom{6}{2}\left(\frac{x}{2}\right)^{6-2}\left(-\frac{2}{x^2}\right)^2+\binom{6}{3}\left(\frac{x}{2}\right)^{6-3}\left(-\frac{2}{x^2}\right)^3$$

$$+\binom{6}{4}\left(\frac{x}{2}\right)^{6-4}\left(-\frac{2}{x^2}\right)^4+\binom{6}{5}\left(\frac{x}{2}\right)^{6-5}\left(-\frac{2}{x^2}\right)^5+\binom{6}{6}\left(\frac{x}{2}\right)^{6-6}\left(-\frac{2}{x^2}\right)^6$$

$$=(1)\left(\frac{x}{2}\right)^6-(6)\left(\frac{x}{2}\right)^5\left(\frac{2}{x^2}\right)+(15)\left(\frac{x}{2}\right)^4\left(\frac{2}{x^2}\right)^2-(20)\left(\frac{x}{2}\right)^3\left(\frac{2}{x^2}\right)^3$$

$$+(15)\left(\frac{x}{2}\right)^2\left(\frac{2}{x^2}\right)^4-(6)\left(\frac{x}{2}\right)^1\left(\frac{2}{x^2}\right)^5+(1)(1)\left(\frac{2}{x^2}\right)^6$$

$$=\left(\frac{x^6}{64}\right)-6\left(\frac{x^5}{32}\right)\left(\frac{2}{x^2}\right)+15\left(\frac{x^4}{16}\right)\left(\frac{4}{x^4}\right)-20\left(\frac{x^3}{8}\right)\left(\frac{8}{x^6}\right)$$

$$+15\left(\frac{x^2}{4}\right)\left(\frac{16}{x^8}\right)-6\left(\frac{x}{2}\right)\left(\frac{32}{x^{10}}\right)+\left(\frac{64}{x^{12}}\right)$$

$$=\frac{x^6}{64}-\frac{3x^3}{8}+\frac{15}{4}-\frac{20}{x^3}+\frac{60}{x^6}-\frac{96}{x^9}+\frac{64}{x^{12}}$$

(iii) *Do yourself*

(iv) *Do yourself*

(v) *Do yourself*

(vi) $\left(\sqrt{\frac{a}{x}}-\sqrt{\frac{x}{a}}\right)^{6}=\binom{6}{0}\left(\sqrt{\frac{a}{x}}\right)^{6}+\binom{6}{1}\left(\sqrt{\frac{a}{x}}\right)^{6-1}\left(-\sqrt{\frac{x}{a}}\right)^{1}+\binom{6}{2}\left(\sqrt{\frac{a}{x}}\right)^{6-2}\left(-\sqrt{\frac{x}{a}}\right)^{2}+\binom{6}{3}\left(\sqrt{\frac{a}{x}}\right)^{6-3}$

$\left(-\sqrt{\frac{x}{a}}\right)^{3}+\binom{6}{4}\left(\sqrt{\frac{a}{x}}\right)^{6-4}\left(-\sqrt{\frac{x}{a}}\right)^{4}+\binom{6}{5}\left(\sqrt{\frac{a}{x}}\right)^{6-5}\left(-\sqrt{\frac{x}{a}}\right)^{5}+\binom{6}{6}\left(\sqrt{\frac{a}{x}}\right)^{6-6}\left(-\sqrt{\frac{x}{a}}\right)^{6}$

$=(1)\left(\sqrt{\frac{a}{x}}\right)^{6}-(6)\left(\sqrt{\frac{a}{x}}\right)^{5}\left(\sqrt{\frac{x}{a}}\right)^{1}+(15)\left(\sqrt{\frac{a}{x}}\right)^{4}\left(\sqrt{\frac{x}{a}}\right)^{2}-(20)\left(\sqrt{\frac{a}{x}}\right)^{3}\left(\sqrt{\frac{x}{a}}\right)^{3}-$

$\left(\sqrt{\frac{x}{a}}\right)^{3}+(15)\left(\sqrt{\frac{a}{x}}\right)^{2}\left(\sqrt{\frac{x}{a}}\right)^{4}-(6)\left(\sqrt{\frac{a}{x}}\right)^{1}\left(\sqrt{\frac{x}{a}}\right)^{5}+(1)\left(\sqrt{\frac{a}{x}}\right)^{0}\left(\sqrt{\frac{x}{a}}\right)^{6}\quad=$

$\left(\sqrt{\frac{a}{x}}\right)^{6}-6\left(\sqrt{\frac{a}{x}}\right)^{5}\left(\sqrt{\frac{a}{x}}\right)^{-1}+15\left(\sqrt{\frac{a}{x}}\right)^{4}\left(\sqrt{\frac{a}{x}}\right)^{-2}-20\left(\sqrt{\frac{a}{x}}\right)^{3}\left(\sqrt{\frac{a}{x}}\right)^{-3}$

$+15\left(\sqrt{\frac{x}{a}}\right)^{-2}\left(\sqrt{\frac{x}{a}}\right)^{4}-6\left(\sqrt{\frac{x}{a}}\right)^{-1}\left(\sqrt{\frac{x}{a}}\right)^{5}+1(1)\left(\sqrt{\frac{x}{a}}\right)^{6}$

$=\left(\sqrt{\frac{a}{x}}\right)^{6}-6\left(\sqrt{\frac{a}{x}}\right)^{5-1}+15\left(\sqrt{\frac{a}{x}}\right)^{4-2}-20\left(\sqrt{\frac{a}{x}}\right)^{3-3}+15\left(\sqrt{\frac{x}{a}}\right)^{-2+4}-6\left(\sqrt{\frac{x}{a}}\right)^{-1+5}+1\left(\sqrt{\frac{x}{a}}\right)^{6}$

$=\left(\sqrt{\frac{a}{x}}\right)^{6}-6\left(\sqrt{\frac{a}{x}}\right)^{4}+15\left(\sqrt{\frac{a}{x}}\right)^{2}-20\left(\sqrt{\frac{a}{x}}\right)^{0}+15\left(\sqrt{\frac{x}{a}}\right)^{2}-6\left(\sqrt{\frac{x}{a}}\right)^{4}+\left(\sqrt{\frac{x}{a}}\right)^{6}$

$=\left(\left(\frac{a}{x}\right)^{\frac{1}{2}}\right)^{6}-6\left(\left(\frac{a}{x}\right)^{\frac{1}{2}}\right)^{4}+15\left(\left(\frac{a}{x}\right)^{\frac{1}{2}}\right)^{2}-20(1)+15\left(\left(\frac{x}{a}\right)^{\frac{1}{2}}\right)^{2}-6\left(\left(\frac{x}{a}\right)^{\frac{1}{2}}\right)^{4}+\left(\left(\frac{x}{a}\right)^{\frac{1}{2}}\right)^{6}$

$=\left(\frac{a}{x}\right)^{3}-6\left(\frac{a}{x}\right)^{2}+15\left(\frac{a}{x}\right)-20+15\left(\frac{x}{a}\right)-6\left(\frac{x}{a}\right)^{2}+\left(\frac{x}{a}\right)^{3}$

$=\frac{a^{3}}{x^{3}}-6\frac{a^{2}}{x^{2}}+15\frac{a}{x}-20+15\frac{x}{a}-6\frac{x^{2}}{a^{2}}+\frac{x^{3}}{a^{3}}$

Question # 2

Calculate the following by means of binomial theorem:

(i) $(0.97)^3$ (ii) $(2.02)^4$ (iii) $(9.98)^4$ (iv) $(2.1)^5$

Solution (i) $(0.97)^3=(1-0.03)^3$

$$=\binom{3}{0}(1)^3+\binom{3}{1}(1)^2(-0.03)+\binom{3}{2}(1)^1(-0.03)^2++\binom{3}{3}(-0.03)^3$$

$$= (1)(1) + 3(1)(-0.03) + 3(1)(0.0009) + +(1)(-0.000024)$$
$$= 1 - 0.09 + 0.0027 - 0.000027 = 0.912673$$

(ii) $(2.02)^4 = (2+0.02)^4$ *Now do yourself.*

(iii) $(9.98)^4 = (10-0.02)^4$

$$= \binom{4}{0}(10)^4 + \binom{4}{1}(10)^3(-0.02) + \binom{4}{2}(10)^2(-0.02)^2 + \binom{4}{3}(10)^1(-0.02)^3$$
$$+ \binom{4}{4}(10)^0(-0.02)^4$$
$$= (1)(10000) + 4(1000)(-0.02) + 6(100)(0.0004) + 4(10)(-0.000008)$$
$$+ (1)(1)(0.00000016)$$
$$= 10000 - 80 + 0.24 - 0.00032 + 0.00000016 = 9920.23968$$

(iv) $(2.1)^5 = (2+0.1)^5$ *Now do yourself.*

Question # 3

Expand and simplify the following:

(i) $\left(a+\sqrt{2}x\right)^4 + \left(a-\sqrt{2}x\right)^4$ (ii) $\left(2+\sqrt{3}\right)^5 + \left(2-\sqrt{3}\right)^5$

(iii) $(2+i)^5 - (2-i)^5$ (iv) $\left(x+\sqrt{x^2-1}\right)^3 + \left(x-\sqrt{x^2-1}\right)^3$

Solution (i) $\left(a+\sqrt{2}x\right)^4 + \left(a-\sqrt{2}x\right)^4$

We take

$$\left(a+\sqrt{2}x\right)^4$$
$$= \binom{4}{0}a^4 + \binom{4}{1}a^3\left(\sqrt{2}x\right)^1 + \binom{4}{2}a^2\left(\sqrt{2}x\right)^2 + \binom{4}{3}a^1\left(\sqrt{2}x\right)^3 + \binom{4}{4}a^0\left(\sqrt{2}x\right)^4$$
$$= (1)a^4 + (4)a^3\left(\sqrt{2}x\right) + (6)a^2\left(2x^2\right) + (4)a\left(2\sqrt{2}x^3\right) + (1)(1)\left(4x^4\right)$$
$$\Rightarrow \left(a+\sqrt{2}x\right)^4 = a^4 + 4\sqrt{2}a^3x + 12a^2x^2 + 8\sqrt{2}\,ax^3 + 4x^4 \;\ldots\ldots\ldots\ldots\; \text{(i)}$$

Replacing $\sqrt{2}$ by $-\sqrt{2}$ in eq. (i)

$$\left(a-\sqrt{2}x\right)^4 = a^4 + 4\left(-\sqrt{2}\right)a^3x + 12a^2x^2 + 8\left(-\sqrt{2}\right)ax^3 + 4x^4$$
$$= a^4 - 4\sqrt{2}a^3x + 12a^2x^2 - 8\sqrt{2}\,ax^3 + 4x^4 \;\ldots\ldots\ldots\ldots\ldots\; \text{(ii)}$$

Adding (i) & (ii)

$$\left(a+\sqrt{2}x\right)^4 + \left(a-\sqrt{2}x\right)^4 = 2a^4 + 24a^2x^2 + 8x^4$$

(ii) *Do yourself.*

(iii) Since

$$(2+i)^5=\binom{5}{0}2^5+\binom{5}{1}2^{5-1}i+\binom{5}{2}2^{5-2}i^2+\binom{5}{3}2^{5-3}i^3+\binom{5}{4}2^{5-4}i^4+\binom{5}{5}2^{5-5}i^5$$

$$=(1)2^5+(5)2^4i+(10)2^3i^2+(10)2^2i^3+(5)2^1i^4+(1)2^0i^5$$

$$=32+80i+80i^2+40i^3+10i^4+i^5 \;\ldots\ldots\ldots\ldots\; \text{(i)}$$

Replacing i by $-i$ in eq. (i)

$$(2+i)^5=32+80(-i)+80(-i)^2+40(-i)^3+10(-i)^4+(-i)^5$$

$$=32-80i+80i^2-40i^3+10i^4-i^5 \;\ldots\ldots\text{(ii)}$$

Subtracting (i) & (ii)

$$(2+i)^5-(2-i)^5 \;=160i+80i^3+2i^5$$

$$=160i+80(-1)\cdot i+2(-1)^2\cdot i$$

$$=160i-80i+2i \;=\; 82i$$

(iv) $\left(x+\sqrt{x^2-1}\right)^3+\left(x+\sqrt{x^2-1}\right)^3$

Suppose $t=\sqrt{x^2-1}$ then

$$\left(x+\sqrt{x^2-1}\right)^3+\left(x+\sqrt{x^2-1}\right)^3=(x+t)^3+(x+t)^3$$

$$=\left((x)^3+3(x)^2(t)+3(x)(t)^2+(t)^3\right)+\left((x)^3+3(x)^2(-t)+3(x)(-t)^2+(-t)^3\right)$$

$$=x^3+3x^2t+3xt^2+t^3+x^3-3x^2t+3xt^2-t^3$$

$$=2x^3+6xt^2$$

$$=2x^3+6x\left(\sqrt{x^2-1}\right)^2 \qquad \because\; t=\sqrt{x^2-1}$$

$$=2x^3+6x\left(x^2-1\right) \;=2x^3+6x^3-6x \;=8x^3-6x$$

Question # 4

Expand the following in ascending powers of x:

(i) $(2+x-x^2)^4$ (ii) $(1-x+x^2)^4$ (iii) $(1-x-x^2)^4$

Solution (i) $(2+x-x^2)^4$

Put $t=2+x$ then

$$(2+x-x^2)^4=(t-x^2)^4$$

$$=\binom{4}{0}(t)^4+\binom{4}{1}(t)^3(-x^2)+\binom{4}{2}(t)^2(-x^2)^2+\binom{4}{3}(t)^1(-x^2)^3+\binom{4}{4}(t)^0(-x^2)^4$$

$$=(1)(t)^4-(4)(t)^3(x^2)+(6)(t)^2(x^4)-(4)(t)(x^6)+(1)(1)(x^8)$$

$$=t^4-4t^3x^2+6t^2x^4-4tx^6+x^8 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; \text{(i)}$$

Now

$$t^4=(2+x)^4=\binom{4}{0}(2)^4+\binom{4}{1}(2)^3(x)+\binom{4}{2}(2)^2(x)^2+\binom{4}{3}(2)^1(x)^3+\binom{4}{4}(2)^0(x)^4$$

$$=(1)(16)+(4)(8)(x)+(6)(4)(x^2)+(4)(2)(x^3)+(1)(1)(x^4)$$

$$=16+32x+24x^2+8x^3+x^4$$

Also

$$t^3 = (2+x)^3 = (2)^3 + (3)(2)^2(x) + (3)(2)^1(x)^2 + (x)^3$$
$$= 8 + 12x + 6x^2 + x^3$$
$$t^2 = (2+x)^2 = 4 + 4x + x^2$$

Putting values of t^4, t^3, t^2 and t in equation (i)

$$(2+x-x^2)^4 = (16+32x+24x^2+8x^3+x^4) - 4(8+12x+6x^2+x^3)x^2$$
$$+6(4+4x+x^2)x^4 - 4(2+x)x^6 + x^8$$
$$= 16+32x+24x^2+8x^3+x^4-32x^2-48x^3-24x^4-4x^5$$
$$+24x^4+24x^5+6x^6-8x^6+4x^7+x^8$$
$$= 16+32-8x^2-40x^3+x^4+20x^5-2x^6-4x^7-x^8$$

(ii) Suppose $t = 1-x$ *Do yourself*

(iii) Suppose $t = 1-x$ *Do yourself*

Question # 5

Expand the following in descending powers of x:

(i) $\left(x^2+x-1\right)^3$ (ii) $\left(x-1-\dfrac{1}{x}\right)^3$

Solution (i) Suppose $t = x-1$ *Do yourself*

(ii) $\left(x-1-\dfrac{1}{x}\right)^3$

Suppose $t = x-1$ then

$$\left(t-\frac{1}{x}\right)^3 = (t)^3 + 3(t)^2\left(-\frac{1}{x}\right) + 3(t)\left(-\frac{1}{x}\right)^2 + \left(-\frac{1}{x}\right)^3$$
$$= t^3 - 3t^2\cdot\frac{1}{x} + 3t\cdot\frac{1}{x^2} - \frac{1}{x^3} \quad \text{........... (i)}$$

Now

$$t^3 = \left(x-1\right)^3 = (x)^3 + 3(x)^2(-1) + 3(x)(-1)^2 + (-1)^3$$
$$= x^3 - 3x^2 + 3x - 1$$
$$t^2 = \left(x-1\right)^2 = x^2 - 2x + 1$$

Putting values of t^3, t^2 and t in equation (i)

$$\left(x-1-\frac{1}{x}\right)^3 = (x^3-3x^2+3x-1) - 3(x^2-2x+1)\cdot\frac{1}{x} + 3(x-1)\cdot\frac{1}{x^2} - \frac{1}{x^3}$$
$$= x^3 - 3x^2 + 3x - 1 - 3x + 6 - 3\frac{1}{x} + 3\frac{1}{x} - 3\frac{1}{x^2} - \frac{1}{x^3}$$
$$= x^3 - 3x^2 + 5 - \frac{3}{x^2} - \frac{1}{x^3}$$

Question # 6

Find the term involving:

(i) x^4 in the expansion of $(3-2x)^7$ (ii) x^{-2} in the expansion of $\left(x-\dfrac{2}{x^2}\right)^{13}$

(iii) a^4 in the expansion of $\left(\dfrac{2}{x}-a\right)^9$ (iv) y^3 in the expansion of $\left(x-\sqrt{y}\right)^{11}$

Solution (i) Since

$$T_{r+1}=\binom{n}{r}a^{n-r}x^r$$

Here $a=3$, $x=-2x$, $n=7$ so

$$T_{r+1}=\binom{7}{r}(3)^{7-r}(-2x)^r=\binom{7}{r}(3)^{7-r}(-2)^r(x)^r$$

For term involving x^4 we must have

$$x^r=x^4 \Rightarrow r=4$$

So

$$T_{4+1}=\binom{7}{4}(3)^{7-4}(-2)^4(x)^4$$

$$\Rightarrow T_5=(35)(3)^3(-2)^4(x)^4=(35)(27)(16)(x)^4$$

$$=15120x^4$$

(ii) Since $T_{r+1}=\binom{n}{r}a^{n-r}x^r$

Here $a=x$, $x=-\dfrac{2}{x^2}$, $n=13$ so

$$T_{r+1}=\binom{13}{r}(x)^{13-r}\left(-\frac{2}{x^2}\right)^r=\binom{13}{r}(x)^{13-r}(-2)^r(x)^{-2r}$$

$$=\binom{13}{r}(x)^{13-r-2r}(-2)^r=\binom{13}{r}(x)^{13-3r}(-2)^r$$

For term involving x^{-2} we must have

$$x^{13-3r}=x^{-2} \Rightarrow 13-3r=-2 \Rightarrow -3r=-2-13$$

$$\Rightarrow -3r=-15 \Rightarrow r=5$$

So

$$T_{5+1}=\binom{13}{5}(x)^{13-3(5)}(-2)^5$$

$$\Rightarrow T_6=(1287)(x)^{13-15}(-32)=-41184x^{-2}$$

(iii) Since

$$T_{r+1}=\binom{n}{r}a^{n-r}x^r$$

Here $a=\dfrac{2}{x}$, $x=-a$, $n=9$ so

$$T_{r+1}=\binom{9}{r}\left(\frac{2}{x}\right)^{9-r}(-a)^r=\binom{9}{r}\left(\frac{2}{x}\right)^{9-r}(-1)^r(a)^r$$

For term involving a^4 we must have

$$a^r=a^4 \;\Rightarrow\; r=4$$

So $$T_{4+1}=\binom{9}{4}\left(\frac{2}{x}\right)^{9-4}(-1)^4(a)^4$$

$$\Rightarrow\; T_5=(126)\left(\frac{2}{x}\right)^5(1)a^4=(126)\left(\frac{32}{x^5}\right)a^4=4032\frac{a^4}{x^5}$$

(iv) Here $a=x$, $x=-\sqrt{y}$, $n=11$ so

Since

$$T_{r+1}=\binom{n}{r}a^{n-r}x^r$$

$$T_{r+1}=\binom{11}{r}(x)^{11-r}\left(-\sqrt{y}\right)^r=\binom{11}{r}(x)^{11-r}\left(-y^{\frac{1}{2}}\right)^r$$

$$=\binom{11}{r}(x)^{11-r}(-1)^r\left(y^{\frac{r}{2}}\right)$$

For term involving y^3 we must have

$$y^{\frac{r}{2}}=y^3 \;\Rightarrow\; \frac{r}{2}=3 \;\Rightarrow\; r=6$$

So $$T_{6+1}=\binom{11}{6}(x)^{11-6}(-1)^6\left(y^{\frac{6}{2}}\right)$$

$$\Rightarrow\; T_7=(462)(x)^5(1)\left(y^3\right)=462x^5y^3$$

Question # 7

Find the coefficient of;

(i) x^5 in the expansion of $\left(x^2-\dfrac{3}{2x}\right)^{10}$ (ii) x^n in the expansion of $\left(x^2-\dfrac{1}{x}\right)^{2n}$

Solution (i) Here $a=x^2$, $x=-\dfrac{3}{2x}$, $n=10$ so

Since

$$T_{r+1}=\binom{n}{r}a^{n-r}x^r$$

$$T_{r+1}=\binom{10}{r}\left(x^2\right)^{10-r}\left(-\frac{3}{2x}\right)^r=\binom{10}{r}(x)^{2(10-r)}(-1)^r\frac{(3)^r}{(2)^r(x)^r}$$

$$=\binom{10}{r}(x)^{20-2r}(-1)^r(3)^r(2)^{-r}(x)^{-r}=\binom{10}{r}(x)^{20-2r-r}(-1)^r(3)^r(2)^{-r}$$

$$=\binom{10}{r}(x)^{20-3r}(-1)^r(3)^r(2)^{-r}$$

For term involving x^5 we must have

$$x^{20-3r}=x^5 \quad\Rightarrow\quad 20-3r=5 \quad\Rightarrow\quad -3r=5-20$$

$$\Rightarrow\quad -3r=-15 \quad\Rightarrow\quad r=5$$

So $\quad T_{5+1}=\binom{10}{5}(x)^{20-3(5)}(-1)^5(3)^5(2)^{-5}$

$$\Rightarrow T_6=252(x)^{20-15}(-1)^5(3)^5\frac{1}{2^5}=-252(x)^5(243)\frac{1}{32}$$

$$=-\frac{61236}{32}x^5=-\frac{15309}{8}x^5$$

Hence coefficient of $x^5=-\dfrac{15309}{8}$

(ii) Here $\quad a=x^2,\;\; x=-\dfrac{1}{x},\;\; n=2n \quad$ so

Since

$$T_{r+1}=\binom{n}{r}a^{n-r}x^r$$

$$T_{r+1}=\binom{2n}{r}(x^2)^{2n-r}\left(-\frac{1}{x}\right)^r=\binom{2n}{r}(x)^{2(2n-r)}(-1)^r\frac{1}{x^r}$$

$$=\binom{2n}{r}(x)^{4n-2r}(-1)^r x^{-r}=\binom{2n}{r}(x)^{4n-2r-r}(-1)^r$$

$$=\binom{2n}{r}(x)^{4n-3r}(-1)^r$$

For term involving x^n we must have

$$x^{4n-3r}=x^n \quad\Rightarrow\quad 4n-3r=n \quad\Rightarrow\quad -3r=n-4n$$

$$\Rightarrow\quad -3r=-3n \quad\Rightarrow\quad r=n$$

So $\quad T_{n+1}=\binom{2n}{n}(x)^{4n-3n}(-1)^n$

$$=\frac{(2n)!}{(2n-n)!\cdot n!}(x)^n(-1)^n=\frac{(2n)!}{n!\cdot n!}(x)^n(-1)^n$$

$$=(-1)^n\frac{(2n)!}{(n!)^2}x^n$$

Hence coefficient of $x^n=(-1)^n\dfrac{(2n)!}{(n!)^2}$

Question # 8

Find $6th$ term in the expansion of $\left(x^2-\dfrac{3}{2x}\right)^{10}$

Solution Here $a = x^2$, $x = -\dfrac{3}{2x}$, $n = 10$ and $r + 1 = 6 \Rightarrow r = 5$ so

Since

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$T_{5+1} = \binom{10}{5} \left(x^2\right)^{10-5} \left(-\frac{3}{2x}\right)^5$$

$$\Rightarrow T_6 = 252\left(x^2\right)^5 \left(-\frac{3^5}{(2x)^5}\right) = 252\,x^{10}\left(-\frac{243}{32x^5}\right)$$

$$= -\frac{61236}{32} x^{10-5} = -\frac{15309}{8} x^5$$

Question # 9

Find the term independent of x in the following expansions..

(i) $\left(x - \dfrac{2}{x}\right)^{10}$ (ii) $\left(\sqrt{x} - \dfrac{1}{2x^2}\right)^{10}$ (iii) $\left(1 + x^2\right)^3 \left(1 + \dfrac{1}{x^2}\right)^4$

Solution (i) *Do yourself as Q # 9* (ii)

(ii) Here $a = \sqrt{x}$, $x = \dfrac{1}{2x^2}$, $n = 10$ so

Since

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$T_{r+1} = \binom{10}{r} \left(\sqrt{x}\right)^{10-r} \left(\frac{1}{2x^2}\right)^r = \binom{10}{r}\left(x^{\frac{1}{2}}\right)^{10-r}\left(\frac{1}{2^r x^{2r}}\right)$$

$$= \binom{10}{r}(x)^{\frac{1}{2}(10-r)} \frac{1}{2^r} x^{-2r} = \binom{10}{r}(x)^{5-\frac{r}{2}} \frac{1}{2^r} x^{-2r}$$

$$= \binom{10}{r}(x)^{5-\frac{r}{2}-2r} \frac{1}{2^r} = \binom{10}{r}(x)^{5-\frac{5r}{2}} \frac{1}{2^r}$$

For term independent of x we must have

$$x^{5-\frac{5r}{2}} = x^0 \Rightarrow 5 - \frac{5r}{2} = 0 \Rightarrow -\frac{5r}{2} = -5$$

$$\Rightarrow r = (-5)\left(-\frac{2}{5}\right) \Rightarrow r = 2$$

So $\quad T_{2+1} = \dbinom{10}{2}(x)^{5-\frac{5(2)}{2}} \dfrac{1}{2^2}$

$$\Rightarrow T_3 = 45\,(x)^{5-5}\,\frac{1}{4} = 45\,x^0\,\frac{1}{4}$$

$$= 45\,(1)\,\frac{1}{4} = \frac{45}{4}$$

(iii) $(1+x^2)^3\left(1+\dfrac{1}{x^2}\right)^4 = (1+x^2)^3\left(\dfrac{x^2+1}{x^2}\right)^4$

$$= (1+x^2)^3\frac{\left(x^2+1\right)^4}{\left(x^2\right)^4} = (1+x^2)^3\frac{\left(1+x^2\right)^4}{x^8}$$

$$= x^{-8}(1+x^2)^{3+4} = x^{-8}(1+x^2)^7$$

Now $\qquad T_{r+1} = x^{-8}\binom{n}{r}a^{n-r}x^r$

Where $\qquad n=7,\ a=1,\ x=x^2$

$$T_{r+1} = x^{-8}\binom{7}{r}(1)^{7-r}\left(x^2\right)^r = x^{-8}\binom{7}{r}(1)\ x^{2r}$$

$$= \binom{7}{r}x^{2r-8}$$

For term independent of x we must have

$$x^{2r-8} = x^0 \Rightarrow 2r-8=0 \Rightarrow 2r=8 \Rightarrow r=4$$

So

$$T_{4+1} = \binom{7}{4}x^{2(4)-8}$$

$$\Rightarrow T_5 = 35\ x^{8-8} = 35\ x^0 = 35$$

Question # 10

Determine the middle term in the following expensions:

(i) $\left(\dfrac{1}{x}-\dfrac{x^2}{2}\right)^{12}$ (ii) $\left(\dfrac{3}{2}x-\dfrac{1}{3x}\right)^{11}$ (iii) $\left(2x-\dfrac{1}{2x}\right)^{2m+1}$

Solution (i) $\qquad \left(\dfrac{1}{x}-\dfrac{x^2}{2}\right)^{12}$

Since $n=12$ is an even so middle terms is $\dfrac{n+2}{2}=\dfrac{12+2}{2}=7$

Therefore $r+1=7 \Rightarrow r=7-1=6$

And $a=\dfrac{1}{x},\ x=-\dfrac{x^2}{2}$ and $n=12$

Now

$$T_{r+1} = \binom{n}{r}a^{n-r}x^r$$

$$\Rightarrow T_{6+1} = \binom{12}{6}\left(\frac{1}{x}\right)^{12-6}\left(-\frac{x^2}{2}\right)^6$$

$$\Rightarrow\ T_7 = 924\,\frac{1}{x^6}\,\frac{x^{12}}{64} \quad = \frac{924}{64}x^{12-6}$$
$$= \frac{231}{16}x^6$$

Thus the middle terms of the given expansion is $\frac{231}{16}x^6$.

(ii) Since $n=11$ is odd so the middle terms are $\frac{n+1}{2}=\frac{11+1}{2}=6$ and

$$\frac{n+3}{2}=\frac{11+3}{2}=7$$

So for first middle term

$$a=\frac{3}{2}x,\quad x=-\frac{1}{3x},\quad n=11 \text{ and } r+1=6 \ \Rightarrow\ r=5$$

Now

$$T_{r+1}=\binom{n}{r}a^{n-r}x^r \qquad \Rightarrow\ T_{5+1}=\binom{11}{5}\left(\frac{3}{2}x\right)^{11-5}\left(-\frac{1}{3x}\right)^5$$

Now simplify yourself.

Now for second middle term

$$r+1=7 \quad \Rightarrow\ r=6$$

so $\quad T_{6+1}=\binom{11}{6}\left(\frac{3}{2}x\right)^{11-6}\left(-\frac{1}{3x}\right)^6$ *Now simplify yourself.*

(iii) Since $n=2m+1$ is odd so there are two middle terms

First middle term = $\frac{n+1}{2}=\frac{2m+1+1}{2}=\frac{2m+2}{2}=m+1$

Second middle terms = $\frac{n+3}{2}=\frac{2m+1+3}{2}=\frac{2m+4}{2}=m+2$

Here $a=2x$, $\quad x=-\frac{1}{2x}$ and $\quad n=2m+1$

For first middle term $\ r+1=m+1 \Rightarrow\ r=m$.

Since

$$T_{r+1}=\binom{n}{r}a^{n-r}x^r$$

$$\Rightarrow\ T_{m+1}=\binom{2m+1}{m}(2x)^{2m+1-m}\left(-\frac{1}{2x}\right)^m = \frac{(2m+1)!}{(2m+1-m)!\cdot m!}(2x)^{m+1}\left(-\frac{1}{2x}\right)^m$$
$$=\frac{(2m+1)!}{(m+1)!\cdot m!}(2)^{m+1}(x)^{m+1}(-1)^m\left(\frac{1}{2}\right)^m\left(\frac{1}{x}\right)^m$$
$$=\frac{(2m+1)!}{(m+1)!\cdot m!}(2)^{m+1}(x)^{m+1}(-1)^m(2)^{-m}(x)^{-m}$$

$$= \frac{(2m+1)!}{(m+1)!\cdot m!}(2)^{m+1-m}(x)^{m+1-m}(-1)^m = \frac{(2m+1)!}{(m+1)!\cdot m!}(2)^1(x)^1(-1)^m$$

$$= \frac{(2m+1)!}{(m+1)!\cdot m!}2x(-1)^m$$

For second middle term

$$r+1 = m+2 \quad \Rightarrow \quad r = m+2-1 \quad \Rightarrow \quad r = m+1$$

As $\quad T_{r+1} = \binom{n}{r} a^{n-r} x^r$

$$\Rightarrow \ T_{m+1+1} = \binom{2m+1}{m+1}(2x)^{(2m+1)-(m+1)}\left(-\frac{1}{2x}\right)^{m+1}$$

Now simplify yourself

Question # 11

Find $(2n+1)th$ term of the end in the expansion of $\left(x - \frac{1}{2x}\right)^{3n}$

Solution Here $a = x$, $\quad x = -\frac{1}{2x}$,

Number of term from the end = $2n$ + 1

To make it from beginning we take $a = -\frac{1}{2x}$, $\quad x = x$ and $\quad r+1 = 2n+1$

$\Rightarrow \ r = 2n$

As $\quad T_{r+1} = \binom{n}{r} a^{n-r} x^r$

$$\Rightarrow \ T_{2n+1} = \binom{3n}{2n}\left(-\frac{1}{2x}\right)^{3n-2n}(x)^{2n} = \frac{(3n)!}{(3n-2n)!\cdot(2n)!}\left(-\frac{1}{2x}\right)^n x^{2n}$$

$$= \frac{(3n)!}{(n)!\cdot(2n)!}(-1)^n\frac{1}{2^n \cdot x^n}x^{2n} = \frac{(3n)!}{n!\cdot(2n)!}(-1)^n\frac{1}{2^n}x^{2n-n}$$

$$= \frac{(-1)^n}{2^n}\frac{(3n)!}{n!\cdot(2n)!}x^n \quad \text{Answer}$$

Note: If there are p term in some expansion and qth term is from the end then the term from the beginning will be $= p - q + 1$.

So in above you can use term from the end = $(3n+1) - (2n+1) + 1 = n+1$

Question # 12

Show that the middle term of $(1+x)^{2n}$ is $\frac{1.3.5...(2n-1)}{n!}2^n x^n$

Solution Since $2n$ is even so the middle term is $\frac{2n+2}{2} = n+1$ and

$$a = 1, \quad x = x, \quad n = 2n, \quad r+1 = n+1 \Rightarrow \quad r = n$$

Now $\quad T_{r+1} = \binom{n}{r} a^{n-r} x^r$

$$\Rightarrow\ T_{n+1}=\binom{2n}{n}(1)^{2n-n}x^n$$

$$\Rightarrow\ T_{n+1}=\frac{(2n)!}{(2n-n)!\cdot n!}(1)^n x^n=\frac{(2n)!}{n!\cdot n!}x^n$$

$$=\frac{2n(2n-1)(2n-2)(2n-3)(2n-4)\cdot\ldots\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{n!\cdot n!}x^n$$

$$=\frac{\left[2n\,(2n-2)(2n-4)\cdot\ldots\cdot 4\cdot 2\right]\left[(2n-1)(2n-3)\cdot\ldots\cdot 5\cdot 3\cdot 1\right]}{n!\cdot n!}x^n$$

$$=\frac{2^n\left[n\,(n-1)(n-2)\cdot\ldots\cdot 2\cdot 1\right]\left[(2n-1)(2n-3)\cdot\ldots\cdot 5\cdot 3\cdot 1\right]}{n!\cdot n!}x^n$$

$$=\frac{2^n\,n!\,\left[(2n-1)(2n-3)\cdot\ldots\cdot 5\cdot 3\cdot 1\right]}{n!\cdot n!}x^n$$

$$=\frac{2^n\left[1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)\right]}{n!}x^n$$

$$=\frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)}{n!}\,2^n x^n$$

Question # 13

Show that:

$$\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots+\binom{n}{n-1}=2^{n-1}$$

Solution Consider

$$(1+x)^n=\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\binom{n}{3}x^3+\binom{n}{4}x^4+\binom{n}{5}x^5+\ldots+\binom{n}{n-1}x^{n-1}+\binom{n}{n}x^n$$

………. (i)

Put $x=1$

$$(1+1)^n=\binom{n}{0}+\binom{n}{1}(1)+\binom{n}{2}(1)^2+\binom{n}{3}(1)^3+\binom{n}{4}(1)^4+\binom{n}{5}(1)^5+\ldots+\binom{n}{n-1}(1)^{n-1}+\binom{n}{n}(1)^n$$

$$\Rightarrow\ 2^n=\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\binom{n}{4}+\binom{n}{5}+\ldots+\binom{n}{n-1}+\binom{n}{n}$$

$$\Rightarrow\ 2^n=\left[\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\ldots+\binom{n}{n}\right]+\left[\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots+\binom{n}{n-1}\right]$$

………. (ii)

Now put $x=-1$ in equation (i)

$$(1-1)^n=\binom{n}{0}+\binom{n}{1}(-1)+\binom{n}{2}(-1)^2+\binom{n}{3}(-1)^3+\binom{n}{4}(-1)^4+\binom{n}{5}(-1)^5+\ldots$$

$$\ldots+\binom{n}{n-1}(-1)^{n-1}+\binom{n}{n}(-1)^n$$

If we consider n is even then

$$\Rightarrow\ (0)^n=\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}+\binom{n}{4}-\binom{n}{5}+\ldots-\binom{n}{n-1}+\binom{n}{n}$$

$$\Rightarrow\ 0=\left[\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\ldots+\binom{n}{n}\right]-\left[\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots+\binom{n}{n-1}\right]$$

$$\Rightarrow\ \left[\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots+\binom{n}{n-1}\right]=\left[\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\ldots+\binom{n}{n}\right]$$

Using it in equation (ii)

$$2^n=\left[\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots+\binom{n}{n-1}\right]+\left[\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots+\binom{n}{n-1}\right]$$

$$\Rightarrow\ 2^n=2\left[\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots+\binom{n}{n-1}\right]$$

$$\Rightarrow\ \frac{2^n}{2}=\left[\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots+\binom{n}{n-1}\right]$$

$$\Rightarrow\ 2^{n-1}=\left[\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots+\binom{n}{n-1}\right]$$

$$\Rightarrow\ \binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots+\binom{n}{n-1}=2^{n-1}$$

Question # 14

Show that:

$$\binom{n}{0}+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+\frac{1}{4}\binom{n}{3}+\ldots+\frac{1}{n+1}\binom{n}{n}=\frac{2^{n+1}-1}{n+1}$$

Solution

$$\text{L.H.S}=\binom{n}{0}+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+\frac{1}{4}\binom{n}{3}+\ldots+\frac{1}{n+1}\binom{n}{n}$$

$$=\left[\binom{n}{0}+\frac{1}{2}\left(\frac{n!}{(n-1)!\cdot 1!}\right)+\frac{1}{3}\left(\frac{n!}{(n-2)!\cdot 2!}\right)+\frac{1}{4}\left(\frac{n!}{(n-3)!\cdot 3!}\right)+\ldots+\frac{1}{n+1}\binom{n}{n}\right]$$

$$=\frac{n+1}{n+1}\left[1+\frac{1}{2}\left(\frac{n!}{(n-1)!\cdot 1!}\right)+\frac{1}{3}\left(\frac{n!}{(n-2)!\cdot 2!}\right)+\frac{1}{4}\left(\frac{n!}{(n-3)!\cdot 3!}\right)+\ldots+\frac{1}{n+1}\cdot 1\right]$$

$$=\frac{1}{n+1}\left[(n+1)+\frac{1}{2}\left(\frac{(n+1)\,n!}{(n-1)!\cdot 1!}\right)+\frac{1}{3}\left(\frac{(n+1)\,n!}{(n-2)!\cdot 2!}\right)+\frac{1}{4}\left(\frac{(n+1)\,n!}{(n-3)!\cdot 3!}\right)+\ldots+\frac{n+1}{n+1}\cdot 1\right]$$

$$=\frac{1}{n+1}\left[(n+1)+\left(\frac{(n+1)!}{(n-1)!\cdot 2\cdot 1!}\right)+\left(\frac{(n+1)!}{(n-2)!\cdot 3\cdot 2!}\right)+\left(\frac{(n+1)!}{(n-3)!\cdot 4\cdot 3!}\right)+\ldots+1\right]$$

$$=\frac{1}{n+1}\left[(n+1)+\left(\frac{(n+1)!}{(n+1-2)!\cdot 2!}\right)+\left(\frac{(n+1)!}{(n+1-3)!\cdot 3!}\right)+\left(\frac{(n+1)!}{(n+1-4)!\cdot 4!}\right)+\ldots+1\right]$$

$$=\frac{1}{n+1}\left[\binom{n+1}{1}+\binom{n+1}{2}+\binom{n+1}{3}+\binom{n+1}{4}+\ldots+\binom{n+1}{n+1}\right]$$

$$=\frac{1}{n+1}\left[-1+1+\binom{n+1}{1}+\binom{n+1}{2}+\binom{n+1}{3}+\binom{n+1}{4}+\ldots+\binom{n+1}{n+1}\right]$$

$$= \frac{1}{n+1}\left[-1+\binom{n+1}{0}+\binom{n+1}{1}+\binom{n+1}{2}+\binom{n+1}{3}+\binom{n+1}{4}+\ldots+\binom{n+1}{n+1}\right]$$

$$= \frac{1}{n+1}\left[-1+2^{n+1}\right]$$

$$= \frac{2^{n+1}-1}{n+1} = \text{R.H.S}$$

Remember

$$\binom{n+1}{0}=1\ ,\quad \binom{n+1}{1}=n+1 \quad \text{and} \quad \binom{n+1}{n+1}=1$$

Book: **Exercise 8.2**
Text Book of Algebra and Trigonometry Class XI
Punjab Textbook Board, Lahore.

Binomial Theorem when n is negative or fraction:

When n is negative or fraction and $|x|<1$ then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

Where the general term of binomial expansion is

$$T_{r+1} = \frac{n(n-1)(n-2)\ldots\big(n-(r-1)\big)}{r!}x^r$$

Question # 1

Expand the following upto 4 times, taking the values of x such that the expansion in each case is valid.

(i) $(1-x)^{\frac{1}{2}}$ (ii) $(1+2x)^{-1}$ (iii) $(1+x)^{-\frac{1}{3}}$ (iv) $(4-3x)^{\frac{1}{2}}$

(v) $(8-2x)^{-1}$ (vi) $(2-3x)^{-2}$ (vii) $\dfrac{(1-x)^{-1}}{(1+x)^2}$ (viii) $\dfrac{\sqrt{(1+2x)}}{(1-x)}$

(ix) $\dfrac{(4+2x)^{\frac{1}{2}}}{(2-x)}$ (x) $(1+x-2x^2)^{\frac{1}{2}}$ (xi) $(1-2x+3x^2)^{\frac{1}{2}}$

Solution

(i)
$$(1-x)^{\frac{1}{2}} = 1 + \frac{1}{2}(-x) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(-x)^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(-x)^3 + \ldots$$
$$= 1 - \frac{1}{2}x + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}x^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3\cdot 2}(-x^3) + \ldots$$
$$= 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 + \ldots$$

(ii) *Do yourself as above*

(iii) *Do yourself as above*

(iv)
$$(4-3x)^{\frac{1}{2}} = \left[4\left(1-\frac{3x}{4}\right)\right]^{\frac{1}{2}} = (4)^{\frac{1}{2}}\left(1-\frac{3x}{4}\right)^{\frac{1}{2}} = 2\left(1-\frac{3x}{4}\right)^{\frac{1}{2}}$$

$$= 2\left[1 + \frac{1}{2}\left(-\frac{3x}{4}\right) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(-\frac{3x}{4}\right)^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\left(-\frac{3x}{4}\right)^3 + \ldots\right]$$

$$=2\left[1-\frac{3x}{8}+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}\left(\frac{9x^2}{16}\right)+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3\cdot 2}\left(-\frac{27x^3}{64}\right)+\ldots\right]$$

$$=2\left[1-\frac{3x}{8}-\frac{1}{8}\left(\frac{9x^2}{16}\right)-\frac{1}{16}\left(\frac{27x^3}{64}\right)+\ldots\right]$$

$$=2\left[1-\frac{3x}{8}-\frac{9x^2}{128}-\frac{27x^3}{1024}+\ldots\right]$$

$$=2-\frac{3x}{4}-\frac{9x^2}{64}-\frac{27x^3}{512}+\ldots$$

(v) $(8-2x)^{\frac{1}{2}}=(8)^{-1}\left(1-\frac{2x}{8}\right)^{-1}=\frac{1}{8}\left(1-\frac{x}{4}\right)^{-1}$ *Now do yourself*

(vi) *Do yourself*

(vii) $\dfrac{(1-x)^{-1}}{(1+x)^2}=(1-x)^{-1}(1+x)^{-2}$

$$=\left(1+(-1)(-x)+\frac{(-1)(-1-1)}{2!}(-x)^2+\frac{(-1)(-1-1)(-1-2)}{3!}(-x)^3+\ldots\right)$$

$$\times\left(1+(-2)(x)+\frac{(-2)(-2-1)}{2!}(x)^2+\frac{(-2)(-2-1)(-2-2)}{3!}(x)^3+\ldots\right)$$

$$=\left(1+x+\frac{(-1)(-2)}{2}(x^2)+\frac{(-1)(-2)(-3)}{3\cdot 2}(-x^3)+\ldots\right)$$

$$\times\left(1-2x+\frac{(-2)(-3)}{2}(x)^2+\frac{(-2)(-3)(-4)}{3\cdot 2}(x)^3+\ldots\right)$$

$$=\left(1+x+x^2+x^3+\ldots\ldots\ldots\ldots\right)\times\left(1-2x+3x^2-4x^3+\ldots\right)$$

$$=1+(x-2x)+(x^2-2x^2+3x^2)+(x^3-2x^3+3x^3-4x^3)+\ldots$$

$$=1-x+2x^2-2x^3+\ldots$$

(viii) *Do yourself as above*

(ix) $\dfrac{(4+2x)^{\frac{1}{2}}}{2-x}=(4+2x)^{\frac{1}{2}}(2-x)^{-1}=(4)^{\frac{1}{2}}\left(1+\frac{2x}{4}\right)^{\frac{1}{2}}(2)^{-1}\left(1-\frac{x}{2}\right)^{-1}$

$$=(4)^{\frac{1}{2}}\left(1+\frac{x}{2}\right)^{\frac{1}{2}}(2)^{-1}\left(1-\frac{x}{2}\right)^{-1}=2\left(1+\frac{x}{2}\right)^{\frac{1}{2}}\frac{1}{2}\left(1-\frac{x}{2}\right)^{-1}=\left(1+\frac{x}{2}\right)^{\frac{1}{2}}\left(1-\frac{x}{2}\right)^{-1}$$

$$=\left(1+\frac{x}{2}\right)^{\frac{1}{2}}\left(1-\frac{x}{2}\right)^{-1}$$

$$=\left(1+\frac{1}{2}\left(\frac{x}{2}\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{x}{2}\right)^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\left(\frac{x}{2}\right)^3+\ldots\right)$$

$$\times\left(1+(-1)\left(-\frac{x}{2}\right)+\frac{(-1)(-1-1)}{2!}\left(-\frac{x}{2}\right)^2+\frac{(-1)(-1-1)(-1-2)}{3!}\left(-\frac{x}{2}\right)^3+\ldots\right)$$

$$=\left(1+\frac{x}{4}+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}\left(\frac{x^2}{4}\right)+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3\cdot 2}\left(\frac{x^3}{8}\right)+\ldots\right)$$

$$\times\left(1+\frac{x}{2}+\frac{(-1)(-2)}{2}\left(\frac{x^2}{4}\right)+\frac{(-1)(-2)(-3)}{3\cdot 2}\left(-\frac{x^3}{8}\right)+\ldots\right)$$

$$=\left(1+\frac{x}{4}-\frac{x^2}{32}+\frac{x^3}{128}+\ldots\right)\times\left(1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{8}+\ldots\right)$$

$$=1+\left(\frac{x}{4}+\frac{x}{2}\right)+\left(-\frac{x^2}{32}+\frac{x^2}{8}+\frac{x^2}{4}\right)+\left(\frac{x^3}{128}-\frac{x^3}{64}+\frac{x^3}{16}+\frac{x^3}{8}\right)+\ldots$$

$$=1+\frac{3x}{4}+\frac{11x^2}{32}+\frac{23x^3}{128}+\ldots$$

(x) $\left(1+x-2x^2\right)^{\frac{1}{2}}=\left(1+(x-2x^2)\right)^{\frac{1}{2}}$

$$=1+\frac{1}{2}(x-2x^2)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(x-2x^2)^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(x-2x^2)^3+\ldots$$

$$=1+\frac{1}{2}(x-2x^2)+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}(x^2-4x^3+4x^4)+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3\cdot 2}$$

$$\left(x^3+3(x)^2(-2x^2)+3(x)(-2x^2)^2-(2x^2)^3\right)+\ldots$$

$$=1+\frac{1}{2}(x-2x^2)-\frac{1}{8}(x^2-4x^3+4x^4)+\frac{1}{16}\left(x^3-6x^4+12x^5-8x^6\right)+\ldots$$

$$=1+\frac{1}{2}x-\frac{2}{2}x^2-\frac{1}{8}x^2-\frac{4}{8}x^3+\frac{4}{8}x^4+\frac{1}{16}x^3-\frac{6}{16}x^4+\frac{12}{16}x^5-\frac{8}{16}x^6+\ldots$$

$$=1+\frac{1}{2}x-x^2-\frac{1}{8}x^2-\frac{1}{2}x^3+\frac{1}{2}x^4+\frac{1}{16}x^3-\frac{3}{8}x^4+\frac{3}{4}x^5-\frac{1}{8}x^6+\ldots$$

$$=1+\frac{1}{2}x-\frac{9}{8}x^2-\frac{9}{16}x^3+\ldots$$

(xi) *Do yourself as above*

Question # 2

Use the Binomial theorem find the value of the following to three places of decimials.

(i) $\sqrt{99}$ (ii) $(0.98)^{\frac{1}{2}}$ (iii) $(1.03)^{\frac{1}{3}}$ (iv) $\sqrt[3]{65}$

(v) $\sqrt[4]{17}$ (vi) $\sqrt[5]{31}$ (vii) $\dfrac{1}{\sqrt[3]{998}}$ (viii) $\dfrac{1}{\sqrt[5]{252}}$

(ix) $\dfrac{\sqrt{7}}{\sqrt{8}}$ (x) $(0.998)^{-\frac{1}{3}}$ (xi) $\dfrac{1}{\sqrt[6]{486}}$ (xii) $(1280)^{\frac{1}{4}}$

Solution

(i)
$$\sqrt{99}=(99)^{\frac{1}{2}}=(100-1)^{\frac{1}{2}}=(100)^{\frac{1}{2}}\left(1-\frac{1}{100}\right)^{\frac{1}{2}}$$
$$=10\left(1+\frac{1}{2}\left(-\frac{1}{100}\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(-\frac{1}{100}\right)^2+\ldots\right)$$
$$=10\left(1-\frac{1}{200}+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}\left(\frac{1}{10000}\right)+\ldots\right)$$
$$=10\left(1-0.005-\frac{1}{8}(0.0001)+\ldots\right)$$
$$=10(1-0.005-0.0000125+\ldots)$$
$$\approx 10(0.9949875)=9.949875$$
$$\approx 9.950$$

(ii) $(0.98)^{\frac{1}{2}}=(1-0.02)^{\frac{1}{2}}$ *Now do yourself*

(iii) $(1.03)^{\frac{1}{3}}=(1+0.03)^{\frac{1}{3}}$ *Now do yourself*

(iv) $\sqrt[3]{65}=(65)^{\frac{1}{3}}=(64-1)^{\frac{1}{3}}=(64)^{\frac{1}{3}}\left(1-\frac{1}{64}\right)^{\frac{1}{3}}$ *Now do yourself*

(v) $\sqrt[4]{17}=(17)^{\frac{1}{4}}=(16-1)^{\frac{1}{4}}=(16)^{\frac{1}{4}}\left(1-\frac{1}{16}\right)^{\frac{1}{4}}$ *Now do yourself*

(vi) $\sqrt[5]{31}=(31)^{\frac{1}{5}}=(32-1)^{\frac{1}{5}}=(32)^{\frac{1}{5}}\left(1-\frac{1}{32}\right)^{\frac{1}{5}}$ *Now do yourself*

(vii) $$\frac{1}{\sqrt[3]{998}}=\frac{1}{(998)^{\frac{1}{3}}}=(998)^{-\frac{1}{3}}=(1000-2)^{-\frac{1}{3}}=(1000)^{-\frac{1}{3}}\left(1-\frac{2}{1000}\right)^{-\frac{1}{3}}$$

$$=\left(10^{3}\right)^{-\frac{1}{3}}\left(1-\frac{1}{500}\right)^{-\frac{1}{3}}$$

$$=\left(\frac{1}{10}\right)\left(1+\left(-\frac{1}{3}\right)\left(-\frac{1}{500}\right)+\frac{-\frac{1}{3}\left(-\frac{1}{3}-1\right)}{2!}\left(-\frac{1}{500}\right)^{2}+\ldots\right)$$

$$=\left(\frac{1}{10}\right)\left(1+\left(\frac{1}{1500}\right)+\frac{-\frac{1}{3}\left(-\frac{4}{3}\right)}{2}\left(\frac{1}{250000}\right)+\ldots\right)$$

$$=\left(\frac{1}{10}\right)\left(1+(0.0006667)+\frac{2}{9}(0.000004)+\ldots\right)$$

$$=\left(\frac{1}{10}\right)(1+0.0006667+0.00000089+\ldots)$$

$$\approx\left(\frac{1}{10}\right)(1.00066759)=0.100066759\quad\approx 0.100\quad \textit{Answer}$$

(viii) $$\frac{1}{\sqrt[5]{252}}=\frac{1}{(252)^{\frac{1}{5}}}=(252)^{-\frac{1}{5}}=(243+9)^{-\frac{1}{5}}=(243)^{-\frac{1}{5}}\left(1+\frac{9}{243}\right)^{-\frac{1}{5}}$$

$$=\left(3^{5}\right)^{-\frac{1}{5}}\left(1+\frac{1}{27}\right)^{-\frac{1}{5}}\qquad \textit{Now do yourself as above}$$

(ix) $$\frac{\sqrt{7}}{\sqrt{8}}=\sqrt{\frac{7}{8}}=\left(\frac{7}{8}\right)^{\frac{1}{2}}=\left(1-\frac{1}{8}\right)^{\frac{1}{2}}$$

$$=1+\frac{1}{2}\left(-\frac{1}{8}\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(-\frac{1}{8}\right)^{2}+\ldots$$

$$=1-\frac{1}{16}+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}\left(\frac{1}{64}\right)+\ldots$$

$$=1-\frac{1}{16}-\frac{1}{8}\left(\frac{1}{64}\right)+\ldots$$

$$=1-\frac{1}{16}-\frac{1}{512}+\ldots$$

$$=1-0.0625-0.00195+\ldots$$

$$\approx 0.93555\quad\approx 0.936\qquad \textit{Answer}$$

(x) $(0.998)^{-\frac{1}{3}} = (1-0.002)^{-\frac{1}{3}}$ *Now do yourself as above*

(xi) $\dfrac{1}{\sqrt[6]{486}} = \dfrac{1}{(486)^{\frac{1}{6}}} = (486)^{-\frac{1}{6}} = (729-243)^{-\frac{1}{6}} = (729)^{-\frac{1}{6}}\left(1-\dfrac{243}{729}\right)^{-\frac{1}{6}}$

$= \left(3^6\right)^{-\frac{1}{6}}\left(1-\dfrac{1}{3}\right)^{-\frac{1}{6}}$ *Now do yourself*

(xii) $(1280)^{\frac{1}{4}} = (1296-16)^{\frac{1}{4}} = (1296)^{\frac{1}{4}}\left(1-\dfrac{16}{1296}\right)^{\frac{1}{4}} = \left(6^4\right)^{\frac{1}{4}}\left(1-\dfrac{1}{81}\right)^{\frac{1}{4}}$

Now do yourself

Question # 3

Find the coefficient of x^n in the expansion of

(i) $\dfrac{1+x^2}{(1+x)^2}$ (ii) $\dfrac{(1+x)^2}{(1-x)^2}$ (iii) $\dfrac{(1+x)^3}{(1-x)^2}$

(iv) $\dfrac{(1+x)^2}{(1-x)^3}$ (v) $\left(1-x+x^2-x^3+\ldots\right)^2$

Solution

(i) $\dfrac{1+x^2}{(1+x)^2} = \left(1+x^2\right)(1+x)^{-2}$

$$= \left(1+x^2\right)\left(1+(-2)(x)+\frac{(-2)(-2-1)}{2!}(x)^2+\frac{(-2)(-2-1)(-2-2)}{3!}(x)^3+\ldots\right)$$

$$= \left(1+x^2\right)\left(1-2x+\frac{(-2)(-3)}{2}x^2+\frac{(-2)(-3)(-4)}{3\cdot 2}x^3+\ldots\right)$$

$$= \left(1+x^2\right)\left(1-2x+3x^2-4x^3+\ldots\right)$$

$$= \left(1+x^2\right)\left(1+(-1)2x+(-1)^2 3x^2+(-1)^3 4x^3+\ldots\right)$$

Following in this way we can write

$$\frac{1+x^2}{(1+x)^2} = \left(1+x^2\right)(1+(-1)2x+(-1)^2 3x^2+(-1)^3 4x^3+\ldots+(-1)^{n-2}(n-1)x^{n-2}+$$

$$(-1)^{n-1}(n)x^{n-1}+(-1)^n(n+1)x^n+\ldots)$$

So taking only terms involving x^n we get

$$(-1)^n(n+1)x^n+(-1)^{n-2}(n-1)x^n$$

$$= (-1)^n(n+1)x^n+(-1)^n(-1)^{-2}(n-1)x^n$$

$$= (-1)^n(n+1)x^n+(-1)^n(n-1)x^n \qquad \because\ (-1)^{-2}=1$$

$$= (n+1+n-1)(-1)^n x^n = (2n)(-1)^n x^n$$

Thus the coefficient of term involving x^n is $(2n)(-1)^n$

(ii)

Hint:

After solving you will get

$$\frac{(1+x)^2}{(1-x)^2}=\left(1+2x+x^2\right)(1+2x+3x^2+4x^3+\ldots+(n-1)x^{n-2}+(n)x^{n-1}+(n+1)x^n+\ldots)$$

Do yourself as above

(iii) $\dfrac{(1+x)^3}{(1-x)^2}=(1+x)^3(1-x)^{-2}$

$$=\left(1+x\right)^3\left(1+(-2)(-x)+\frac{(-2)(-2-1)}{2!}(-x)^2+\frac{(-2)(-2-1)(-2-2)}{3!}(-x)^3+\ldots\right)$$

$$=\left(1+x\right)^3\left(1+2x+\frac{(-2)(-3)}{2}(x)^2+\frac{(-2)(-3)(-4)}{3\cdot 2}(-x^3)+\ldots\right)$$

$$=\left(1+3x+3x^2+x^3\right)\left(1+2x+3x^2+4x^3+\ldots\right)$$

Following in this way we can write

$$\frac{(1+x)^3}{(1-x)^2}=\left(1+3x+3x^2+x^3\right)(1+2x+3x^2+4x^3+\ldots+(n-2)x^{n-3}+(n-1)x^{n-2}$$
$$+(n)x^{n-1}+(n+1)x^n+\ldots)$$

So taking only terms involving x^n we have term

$$(n+1)x^n+3(n)x^n+3(n-1)x^n+(n-2)x^n$$
$$=\left((n+1)+3(n)+3(n-1)+(n-2)\right)x^n$$
$$=\left(n+1+3n+3n-3+n-2\right)x^n$$
$$=\left(8n-4\right)x^n$$

Thus the coefficient of term involving x^n is $\left(8n-4\right)$.

(iv) $\dfrac{(1+x)^2}{(1-x)^3}=\left(1+x\right)^2\left(1-x\right)^{-3}$

$$=\left(1+x\right)^2\left(1+(-3)(-x)+\frac{(-3)(-3-1)}{2!}(-x)^2+\frac{(-3)(-3-1)(-3-2)}{3!}(-x)^3+\ldots\right)$$

$$=\left(1+x\right)^2\left(1+(-3)(-x)+\frac{(-3)(-4)}{2}(-x)^2+\frac{(-3)(-4)(-5)}{3\cdot 2}(-x)^3+\ldots\right)$$

$$=\left(1+2x+x^2\right)\left(1+3x+\frac{(3)(4)}{2}(x^2)+\frac{(4)(5)}{2}(x^3)+\ldots\right)$$

$$=\left(1+2x+x^2\right)\left(1+\frac{(2)(3)}{2}x+\frac{(3)(4)}{2}x^2+\frac{(4)(5)}{2}x^3+\ldots\right)$$

Following in this way we can write

$$\frac{(1+x)^2}{(1-x)^3}=\left(1+2x+x^2\right)\left(1+\frac{(2)(3)}{2}x+\frac{(3)(4)}{2}x^2+\frac{(4)(5)}{2}x^3+\ldots\right.$$

$$\left.+\frac{(n-1)(n)}{2}x^{n-2}+\frac{(n)(n+1)}{2}x^{n-1}+\frac{(n+1)(n+2)}{2}x^n+\ldots\right)$$

So taking only terms involving x^n we have term

$$\frac{(n+1)(n+2)}{2}x^n+2\frac{(n)(n+1)}{2}x^n+\frac{(n-1)(n)}{2}x^n$$

$$=\left((n+1)(n+2)+2(n)(n+1)+(n-1)(n)\right)\frac{x^n}{2}$$

$$=\left(n^2+n+2n+2+2n^2+2n+n^2-n\right)\frac{x^n}{2}$$

$$=\left(4n^2+4n+2\right)\frac{x^n}{2}\quad=2\left(2n^2+2n+1\right)\frac{x^n}{2}$$

$$=\left(2n^2+2n+1\right)x^n$$

Thus the coefficient of term involving x^n is $\left(2n^2+2n+1\right)$.

(v) Since we know that

$$(1+x)^{-1}=1-x+x^2-x^3+\ldots$$

Therefore

$$\left(1-x+x^2-x^3+\ldots\right)^2=\left((1+x)^{-1}\right)^2\ =(1+x)^{-2}$$

$$=1+(-2)(x)+\frac{(-2)(-2-1)}{2!}(x)^2+\frac{(-2)(-2-1)(-2-2)}{3!}(x)^3+\ldots$$

$$=1-2x+\frac{(-2)(-3)}{2}(x)^2+\frac{(-2)(-3)(-4)}{3\cdot 2}(x)^3+\ldots$$

$$=1-2x+3x^2-4x^3+\ldots$$

$$=1+(-1)2x+(-1)^2 3x^2(-1)^3 4x^3+\ldots$$

Following in this way we can write

$$=1+(-1)2x+(-1)^2 3x^2(-1)^3 4x^3+\ldots+(-1)^n(n+1)x^n+\ldots$$

So the term involving x^n = $(-1)^n(n+1)x^n$

And hence coefficient of term involving x^n is $(-1)^n(n+1)$

Question # 4

If x so small that its square and higher powers can be neglected, then show that

(i) $\dfrac{1-x}{\sqrt{1+x}}\approx=1-\dfrac{3}{2}x$ (ii) $\dfrac{\sqrt{1+2x}}{\sqrt{1-x}}\approx 1+\dfrac{3}{2}x$

(iii) $\dfrac{(9+7x)^{\frac{1}{2}}-(16+3x)^{\frac{1}{4}}}{4+5x}\approx\dfrac{1}{4}-\dfrac{17}{384}x$ (iv) $\dfrac{\sqrt{4+x}}{(1-x)^3}\approx 2+\dfrac{25}{4}x$

(v) $\dfrac{(1+x)^{\frac{1}{2}}(4-3x)^{\frac{1}{4}}}{(8+5x)^{\frac{1}{3}}} \approx \left(1-\dfrac{5x}{6}\right)$ (vi) $\dfrac{(1-x)^{\frac{1}{2}}(9-4x)^{\frac{1}{2}}}{(8+3x)^{\frac{1}{3}}} \approx \dfrac{3}{2}-\dfrac{61}{48}x$

(vii) $\dfrac{\sqrt{4-x}+(8-x)^{\frac{1}{3}}}{(8-x)^{\frac{1}{3}}} \approx 2-\dfrac{1}{12}x$

Solution

(i)

$$\text{L.H.S} = \frac{1-x}{\sqrt{1+x}} = \frac{1-x}{(1+x)^{\frac{1}{2}}} = (1-x)(1+x)^{-\frac{1}{2}}$$

$$= (1-x)\left(1+\left(-\frac{1}{2}\right)(x) + \textit{squares and higher power of } x\right).$$

$$= 1-x-\frac{1}{2}x + \textit{squares and higher power of } x$$

$$\approx 1-\frac{3}{2}x = \text{R.H.S} \quad \text{Proved}$$

(ii) Since $\dfrac{\sqrt{1+2x}}{\sqrt{1-x}} = (1+2x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$

Now $(1+2x)^{\frac{1}{2}} = 1+\left(\dfrac{1}{2}\right)(2x) +$ squares and higher power of x.

$$\approx 1+x$$

Now $(1-x)^{-\frac{1}{2}} = 1+\left(-\dfrac{1}{2}\right)(-x) +$ squares and higher power of x.

$$\approx 1+\frac{1}{2}x$$

$$\frac{\sqrt{1+2x}}{\sqrt{1-x}} \approx (1+x)\left(1+\frac{1}{2}x\right)$$

$$= 1+x+\frac{1}{2}x$$ ignoring term involving x^2.

$$= 1+\frac{3}{2}x \quad \text{Proved.}$$

(iii) $\dfrac{(9+7x)^{\frac{1}{2}}-(16+3x)^{\frac{1}{4}}}{4+5x} = \left((9+7x)^{\frac{1}{2}}-(16+3x)^{\frac{1}{4}}\right)(4+5x)^{-1}$

Now $(9+7x)^{\frac{1}{2}} = 9^{\frac{1}{2}}\left(1+\dfrac{7x}{9}\right)^{\frac{1}{2}}$

$$= (3^2)^{\frac{1}{2}}\left(1+\left(\frac{1}{2}\right)\left(\frac{7x}{9}\right) + \textit{squres and higher of } x.\right)$$

$$\approx 3\left(1+\frac{7x}{18}\right) = 3+3\left(\frac{7x}{18}\right) = 3+\frac{7x}{6}$$

$$(16+3x)^{\frac{1}{4}} = (16)^{\frac{1}{4}}\left(1+\frac{3x}{16}\right)^{\frac{1}{4}}$$

$$= (2^4)^{\frac{1}{4}}\left(1+\left(\frac{1}{4}\right)\left(\frac{3x}{16}\right)+\textit{square and higher power of } x\right)$$

$$\approx (2)\left(1+\frac{3x}{64}\right) = 2+2\left(\frac{3x}{64}\right) = 2+\frac{3x}{32}$$

$$(4+5x)^{-1} = 4^{-1}\left(1+\frac{5}{4}x\right)^{-1}$$

$$= \frac{1}{4}\left(1+(-1)\left(\frac{5}{4}x\right)+\textit{squares and higher power of } x\right)$$

$$\approx \frac{1}{4}\left(1-\frac{5}{4}x\right) = \frac{1}{4}-\frac{5}{16}x$$

So $$\frac{(9+7x)^{\frac{1}{2}}-(16+3x)^{\frac{1}{4}}}{4+5x} \approx \left[\left(3+\frac{7x}{6}\right)-\left(2+\frac{3x}{32}\right)\right]\left(\frac{1}{4}-\frac{5}{16}x\right)$$

$$= \left[3+\frac{7x}{6}-2-\frac{3x}{32}\right]\left(\frac{1}{4}-\frac{5}{16}x\right) = \left(1+\frac{103}{96}x\right)\left(\frac{1}{4}-\frac{5}{16}x\right)$$

$$= \frac{1}{4}+\frac{103}{384}x-\frac{5}{16}x \quad = \frac{1}{4}-\frac{17}{384}x \quad \text{Proved}$$

(iv) *Do yourself*

(v) $$\frac{(1+x)^{\frac{1}{2}}(4-3x)^{\frac{3}{2}}}{(8+5x)^{\frac{1}{3}}} = (1+x)^{\frac{1}{2}}(4-3x)^{\frac{3}{2}}(8+5x)^{-\frac{1}{3}}$$

Now $$(1+x)^{\frac{1}{2}} = 1+\left(\frac{1}{2}\right)(x)+\textit{square and higher power of } x$$

$$\approx 1+\frac{1}{2}x$$

$$(4-3x)^{\frac{3}{2}} = 4^{\frac{3}{2}}\left(1-\frac{3}{4}x\right)^{\frac{3}{2}}$$

$$= \left(2^2\right)^{\frac{3}{2}}\left(1+\left(\frac{3}{2}\right)\left(-\frac{3}{4}x\right)+\textit{square and higher power of } x\right)$$

$$\approx (2)^3\left(1-\frac{9}{8}x\right) = 8\left(1-\frac{9}{8}x\right)$$

$$(8+5x)^{-\frac{1}{3}} = (8)^{-\frac{1}{3}}\left(1+\frac{5}{8}x\right)^{-\frac{1}{3}}$$

$$=\left(2^3\right)^{-\frac{1}{3}}\left(1+\left(-\frac{1}{3}\right)\left(\frac{5}{8}x\right)+square\ and\ higher\ power\ of\ x\right)$$

$$\approx(2)^{-1}\left(1-\frac{5}{24}x\right)\quad=\frac{1}{2}\left(1-\frac{5}{24}x\right)$$

So $$\frac{(1+x)^{\frac{1}{2}}(4-3x)^{\frac{3}{2}}}{(8+5x)^{\frac{1}{3}}}\approx\left(1+\frac{1}{2}x\right)8\left(1-\frac{9}{8}x\right)\frac{1}{2}\left(1-\frac{5}{24}x\right)$$

$$=\frac{8}{2}\left(1+\frac{1}{2}x\right)\left(1-\frac{9}{8}x-\frac{5}{24}x\right)$$

$$=4\left(1+\frac{1}{2}x\right)\left(1-\frac{4}{3}x\right)\ =4\left(1+\frac{1}{2}x-\frac{4}{3}x\right)\ =4\left(1-\frac{5}{6}x\right)\quad \text{Proved}$$

(vi) *Do yourself as above*
(vii) *Same as Question #*4 (iii)

Question # 5

If x is so small that its cube and higher power can be neglected, then show that

(i) $\sqrt{1-x-2x^2}=1-\frac{1}{2}x-\frac{9}{8}x^2$ (ii) $\sqrt{\frac{1+x}{1-x}}=1+x+\frac{1}{2}x^2$

Solution

(i) $\sqrt{1-x-2x^2}=\left(1-(x+2x^2)\right)^{\frac{1}{2}}$

$$=1+\left(\frac{1}{2}\right)\left(-(x+2x^2)\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(-(x+2x^2)\right)^2+cube\ \&\ higher\ power\ of\ x.$$

$$\approx1-\left(\frac{1}{2}\right)(x+2x^2)+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}(x+2x^2)^2$$

$$\approx1-\frac{1}{2}x-\frac{1}{2}(2x^2)-\frac{1}{8}x^2\quad=1-\frac{1}{2}x-x^2-\frac{1}{8}x^2$$

$$=1-\frac{1}{2}x-\frac{9}{8}x^2\qquad \text{Proved}$$

(ii)

$$\sqrt{\frac{1+x}{1-x}}=\frac{(1+x)^{\frac{1}{2}}}{(1-x)^{\frac{1}{2}}}=(1+x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$$

Now

$$(1+x)^{\frac{1}{2}}=1+\left(\frac{1}{2}\right)x+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2+cube\ \&\ higher\ power\ of\ x.$$

$$\approx1+\frac{1}{2}x+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}x^2\ =1+\frac{1}{2}x-\frac{1}{8}x^2$$

$$(1-x)^{-\frac{1}{2}}=1+\left(-\frac{1}{2}\right)(-x)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}(-x)^2+cube\ \&\ higher\ power\ of\ x.$$

$$\approx 1+\frac{1}{2}x+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}x^2\ =1+\frac{1}{2}x+\frac{3}{8}x^2$$

So

$$\sqrt{\frac{1+x}{1-x}}=\left(1+\frac{1}{2}x-\frac{1}{8}x^2\right)\left(1+\frac{1}{2}x+\frac{3}{8}x^2\right)$$

$$=1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{2}x+\frac{1}{4}x^2+\frac{3}{8}x^2\ =1+x+\frac{1}{2}x^2 \quad \text{Proved}$$

Question # 6

If x is very nearly equal 1, then prove that $px^p-qx^q=(p-q)x^{p+q}$

Solution

Since x is nearly equal to 1 so suppose $x=1+h$,
where h is so small that its square and higher powers be neglected

L.H.S = px^p-qx^q

$$=p(1+h)^p-q(1+h)^q$$
$$=p(1+ph+square\ \&\ higher\ power\ of\ x)$$
$$-q(1+qh+square\ \&\ higher\ power\ of\ h)$$
$$\approx p(1+ph)-q(1+qh)$$
$$=p+p^2h-q-q^2h \ \ldots\ldots\ldots\ldots\ldots\ \text{(i)}$$

Now R.H.S = $(p-q)x^{p+q}$

$$=(p-q)(1+h)^{p+q}$$
$$=(p-q)\big(1+(p+q)h+square\ \&\ higher\ power\ of\ h\big)$$
$$\approx(p-q)\big(1+(p+q)h\big)\ \ =(p-q)\big(1+ph+qh\big)$$
$$=p+p^2h+pqh-q-pqh-q^2h$$
$$=p+p^2h-q-q^2h\ \ldots\ldots\ldots\ldots\ldots\ \text{(ii)}$$

From (i) and (ii)

L.H.S ≈ R.H.S Proved

Question # 7

If $p-q$ is small when compared with p or q, show that

$$\frac{(2n+1)p+(2n-1)q}{(2n-1)p+(2n+1)q}=\left(\frac{p+q}{2q}\right)^{\frac{1}{n}}.$$

Solution Since $p-q$ is small when compare

Therefore let $p-q=h \ \Rightarrow\ p=q+h$

$$\text{L.H.S}=\frac{(2n+1)p+(2n-1)q}{(2n-1)p+(2n+1)q}\ \ =\frac{(2n+1)(q+h)+(2n-1)q}{(2n-1)(q+h)+(2n+1)q}$$

$$=\frac{2nq+q+2nh+h+2nq-q}{2nq-q+2nh-h+2nq+q} \quad =\frac{4nq+2nh+h}{4nq+2nh-h}$$

$$=\frac{4nq+2nh+h}{4nq\left(1+\frac{2nh-h}{4nq}\right)} \quad =\frac{4nq+2nh+h}{4nq}\left(1+\frac{2nh-h}{4nq}\right)^{-1}$$

$$=\frac{4nq+2nh+h}{4nq}\left(1+(-1)\left(\frac{2nh-h}{4nq}\right)+square\ \&\ higher\ power\ of\ x^2\right)$$

$$=\frac{4nq+2nh+h}{4nq}\left(1-\frac{2nh-h}{4nq}\right) \quad =\frac{4nq+2nh+h}{4nq}\left(\frac{4nq-2nh+h}{4nq}\right)$$

$$\approx\frac{16n^2q^2+8n^2hq+4nhq-8n^2hq+4nhq}{16n^2q^2} \qquad \text{ignoring squares of } h$$

$$=\frac{16n^2q^2+8nhq}{16n^2q^2} \quad =\frac{16n^2q^2}{16n^2q^2}+\frac{8nhq}{16n^2q^2}$$

$$=1+\frac{h}{2nq} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

Now R.H.S = $\left(\frac{p+q}{2q}\right)^{\frac{1}{n}}=\left(\frac{q+h+q}{2q}\right)^{\frac{1}{n}}$

$$=\left(\frac{2q+h}{2q}\right)^{\frac{1}{n}}=\left(\frac{2q}{2q}+\frac{h}{2q}\right)^{\frac{1}{n}}=\left(1+\frac{h}{2q}\right)^{\frac{1}{n}}$$

$$=1+\left(\frac{1}{n}\right)\left(\frac{h}{2q}\right)+square\ \&\ higher\ power\ of\ h.$$

$$\approx 1+\frac{h}{2nq} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

Form (*i*) and (*ii*)

L.H.S $\approx$ R.H.S   Proved

Question # 8

Show that $\left(\frac{n}{2(n+N)}\right)^{\frac{1}{2}} \approx \frac{8n}{9n-N}-\frac{n+N}{4n}$ where n and N are nearly equal.

Solution Since n and N are nearly equal therefore consider $N=n+h$, where h is so small that its squares and higher power be neglected.

L.H.S $=\left(\frac{n}{2(n+N)}\right)^{\frac{1}{2}}=\left(\frac{n}{2(n+n+h)}\right)^{\frac{1}{2}}$

$$=\left(\frac{n}{2(2n+h)}\right)^{\frac{1}{2}}=\left(\frac{2(2n+h)}{n}\right)^{-\frac{1}{2}}=\left(\frac{4n+2h}{n}\right)^{-\frac{1}{2}}=\left(4+\frac{2h}{n}\right)^{-\frac{1}{2}}$$

$$= (4)^{-\frac{1}{2}}\left(1+\frac{2h}{4n}\right)^{-\frac{1}{2}} = (2^2)^{-\frac{1}{2}}\left(1+\frac{h}{2n}\right)^{-\frac{1}{2}}$$

$$= (2)^{-1}\left(1+\left(-\frac{1}{2}\right)\frac{h}{2n} + \textit{square \& higher power of } h\right)$$

$$= \frac{1}{2}\left(1-\frac{h}{4n}\right) \quad = \frac{1}{2} - \frac{h}{8n} \ \ldots\ldots\ldots\ldots\ldots\ldots\ (i)$$

Now R.H.S = $\dfrac{8n}{9n-N} - \dfrac{n+N}{4n}$

$$= \frac{8n}{9n-(n+h)} - \frac{n+(n+h)}{4n} \quad = \frac{8n}{9n-n-h} - \frac{n+n+h}{4n}$$

$$= \frac{8n}{8n-h} - \frac{2n+h}{4n} \quad = \frac{8n}{8n\left(1-\frac{h}{8n}\right)} - \frac{2n+h}{4n} \quad = \left(1-\frac{h}{8n}\right)^{-1} - \frac{2n+h}{4n}$$

$$= \left(1+(-1)\left(-\frac{h}{8n}\right) + \textit{square \& higher power of } h\right) - \left(\frac{2n}{4n}+\frac{h}{4n}\right)$$

$$= \left(1+\frac{h}{8n}\right) - \left(\frac{1}{2}+\frac{h}{4n}\right) \quad = 1 + \frac{h}{8n} - \frac{1}{2} - \frac{h}{4n}$$

$$= \frac{1}{2} - \frac{h}{8n} \ \ldots\ldots\ldots\ldots\ldots\ldots\ (ii)$$

From (i) and (ii)

L.H.S = R.H.S Proved

Question # 9

Identify the following series as binomial expansion and find the sum in each case.

(i) $1-\dfrac{1}{2}\left(\dfrac{1}{4}\right)+\dfrac{1\cdot 3}{2!\cdot 4}\left(\dfrac{1}{4}\right)^2-\dfrac{1\cdot 3\cdot 5}{3!\cdot 8}\left(\dfrac{1}{4}\right)^3+\ldots$

(ii) $1-\dfrac{1}{2}\left(\dfrac{1}{2}\right)+\dfrac{1\cdot 3}{2\cdot 4}\left(\dfrac{1}{2}\right)^2-\dfrac{1\cdot 3\cdot 5}{2.4.6}\left(\dfrac{1}{2}\right)^3+\ldots$

(iii) $1+\dfrac{3}{4}+\dfrac{3\cdot 5}{4\cdot 8}+\dfrac{3\cdot 5\cdot 7}{4\cdot 8\cdot 12}+\ldots$

(iv) $1-\dfrac{1}{2}\left(\dfrac{1}{3}\right)+\dfrac{1\cdot 3}{2\cdot 4}\left(\dfrac{1}{3}\right)^2-\dfrac{1\cdot 3\cdot 5}{2.4.6}\left(\dfrac{1}{3}\right)^3+\ldots$

Solution

(i) $$1-\frac{1}{2}\left(\frac{1}{4}\right)+\frac{1\cdot 3}{2!\cdot 4}\left(\frac{1}{4}\right)^2-\frac{1\cdot 3\cdot 5}{3!\cdot 8}\left(\frac{1}{4}\right)^3+\ldots$$

Suppose the given series be identical with

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots$$

This implies $\quad nx = -\dfrac{1}{2}\left(\dfrac{1}{4}\right) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (i)$

$$\frac{n(n-1)}{2!}x^2=\frac{1\cdot 3}{2!\cdot 4}\left(\frac{1}{4}\right)^2 \text{ .... (ii)}$$

From (i) $\quad nx=-\dfrac{1}{8} \Rightarrow x=-\dfrac{1}{8n}$ .... (iii)

Putting value of x in (ii)

$$\frac{n(n-1)}{2!}\left(-\frac{1}{8n}\right)^2=\frac{1\cdot 3}{2!\cdot 4}\left(\frac{1}{4}\right)^2$$

$$\Rightarrow \frac{n(n-1)}{2}\left(\frac{1}{64n^2}\right)=\frac{3}{2\cdot 4}\left(\frac{1}{16}\right)$$

$$\Rightarrow \frac{(n-1)}{128n}=\frac{3}{128} \quad \Rightarrow (n-1)=\frac{3}{128}\cdot 128n \quad \Rightarrow n-1=3n$$

$$\Rightarrow n-3n=1 \quad \Rightarrow -2n=1 \quad \Rightarrow \boxed{n=-\frac{1}{2}}$$

Putting value of n in equation (iii)

$$x=-\frac{1}{8\left(-\frac{1}{2}\right)} \quad \Rightarrow \boxed{x=\frac{1}{4}}$$

So

$$(1+x)^n=\left(1+\frac{1}{4}\right)^{-\frac{1}{2}}=\left(\frac{5}{4}\right)^{-\frac{1}{2}}=\left(\frac{4}{5}\right)^{\frac{1}{2}}=\sqrt{\frac{4}{5}}$$

(ii) *Do yourself as above*

(iii) $\quad 1+\dfrac{3}{4}+\dfrac{3\cdot 5}{4\cdot 8}+\dfrac{3\cdot 5\cdot 7}{4\cdot 8\cdot 12}+\ldots$

Suppose the given series be identical with

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\ldots$$

This implies $\quad nx=\dfrac{3}{4}$ .... (i)

$$\frac{n(n-1)}{2!}x^2=\frac{3\cdot 5}{4\cdot 8} \text{ .... (ii)}$$

From (i) $\quad nx=\dfrac{3}{4} \Rightarrow x=\dfrac{3}{4n}$ .... (iii)

Putting value of x in (ii)

$$\frac{n(n-1)}{2!}\left(\frac{3}{4n}\right)^2=\frac{3\cdot 5}{4\cdot 8} \quad \Rightarrow \frac{n(n-1)}{2}\left(\frac{9}{16n^2}\right)=\frac{15}{32}$$

$$\Rightarrow \frac{9(n-1)}{32n}=\frac{15}{32} \quad \Rightarrow 9(n-1)=\frac{15}{32}\cdot 32n \quad \Rightarrow 9n-9=15n$$

$$\Rightarrow\ 9n-15n=9 \quad \Rightarrow\ -6n=9 \quad \Rightarrow\ n=-\frac{9}{6} \quad \Rightarrow\ \boxed{n=-\frac{3}{2}}$$

Putting value of n in equation (iii)

$$x=-\frac{3}{4\left(-\frac{3}{2}\right)} \Rightarrow \boxed{x=-\frac{1}{2}}$$

So $\quad (1+x)^n=\left(1-\frac{1}{2}\right)^{-\frac{3}{2}}=\left(\frac{1}{2}\right)^{-\frac{3}{2}}=(2)^{\frac{3}{2}}=\left(\sqrt{2}\right)^3=2\sqrt{2}$ Answer

(iv) *Do yourself as above*

Question # 10

Use binomial theorem to show that $1+\frac{1}{4}+\frac{1\cdot 3}{4\cdot 8}+\frac{1\cdot 3\cdot 5}{4\cdot 8\cdot 12}+\ldots=\sqrt{2}$

Solution $\quad 1+\frac{1}{4}+\frac{1\cdot 3}{4\cdot 8}+\frac{1\cdot 3\cdot 5}{4\cdot 8\cdot 12}+\ldots$

Suppose the given series be identical with

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\ldots$$

This implies

$$nx=\frac{1}{4} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (i)$$

$$\frac{n(n-1)}{2!}x^2=\frac{1\cdot 3}{4\cdot 8} \ \ldots\ldots\ldots\ldots\ (ii)$$

From (i) $\quad nx=\frac{1}{4} \ \Rightarrow\ x=\frac{1}{4n} \ \ldots\ldots\ldots\ldots\ (iii)$

Putting value of x in (ii)

$$\frac{n(n-1)}{2!}\left(\frac{1}{4n}\right)^2=\frac{1\cdot 3}{4\cdot 8} \qquad \Rightarrow\ \frac{n(n-1)}{2}\left(\frac{1}{16n^2}\right)=\frac{3}{32}$$

$$\Rightarrow\ \frac{(n-1)}{32n}=\frac{3}{32} \quad \Rightarrow\ (n-1)=\frac{3}{32}\cdot 32n \quad \Rightarrow\ n-1=3n$$

$$\Rightarrow\ n-3n=1 \quad \Rightarrow\ -2n=1 \quad \Rightarrow\ \boxed{n=-\frac{1}{2}}$$

Putting value of n in equation (iii)

$$x=\frac{1}{4\left(-\frac{1}{2}\right)} \Rightarrow \boxed{x=-\frac{1}{2}}$$

So $\quad (1+x)^n=\left(1-\frac{1}{2}\right)^{-\frac{1}{2}}=\left(\frac{1}{2}\right)^{-\frac{1}{2}}=(2)^{\frac{1}{2}}=\sqrt{2}$

Hence $\quad 1+\frac{1}{4}+\frac{1\cdot 3}{4\cdot 8}+\frac{1\cdot 3\cdot 5}{4\cdot 8\cdot 12}+\ldots=\sqrt{2}$ $\qquad$ Proved

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Question # 11

If $y=\frac{1}{3}+\frac{1\cdot 3}{2!}\left(\frac{1}{3}\right)^2+\frac{1\cdot 3\cdot 5}{3!}\left(\frac{1}{3}\right)^3+\ldots$, then prove that $y^2+2y-2=0$.

Solution $\quad y=\frac{1}{3}+\frac{1\cdot 3}{2!}\left(\frac{1}{3}\right)^2+\frac{1\cdot 3\cdot 5}{3!}\left(\frac{1}{3}\right)^3+\ldots$

Adding 1 on both sides

$$1+y=1+\frac{1}{3}+\frac{1\cdot 3}{2!}\left(\frac{1}{3}\right)^2+\frac{1\cdot 3\cdot 5}{3!}\left(\frac{1}{3}\right)^3+\ldots$$

Let the given series be identical with

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\ldots$$

This implies

$$nx=\frac{1}{3}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

$$\frac{n(n-1)}{2!}x^2=\frac{1\cdot 3}{2!}\left(\frac{1}{3}\right)^2\ \ldots\ldots\ldots\ldots \text{(ii)}$$

From (i) $\quad nx=\frac{1}{3}\quad\Rightarrow\quad x=\frac{1}{3n}\ \ldots\ldots\ldots\ldots\ldots \text{(iii)}$

Putting value of x in (ii)

$$\frac{n(n-1)}{2!}\left(\frac{1}{3n}\right)^2=\frac{1\cdot 3}{2!}\left(\frac{1}{3}\right)^2$$

$$\Rightarrow\ \frac{n(n-1)}{2}\left(\frac{1}{9n^2}\right)=\frac{3}{2}\cdot\frac{1}{9}$$

$$\Rightarrow\ \frac{n-1}{18n}=\frac{1}{6}\quad\Rightarrow\quad n-1=\frac{1}{6}\cdot 18\,n$$

$$\Rightarrow\ n-1=3n\quad\Rightarrow\quad n-3n=1$$

$$\Rightarrow\ -2n=1\quad\Rightarrow\quad \boxed{n=-\frac{1}{2}}$$

Putting value of n in equation (iii)

$$x=\frac{1}{3\left(-\frac{1}{2}\right)}\quad\Rightarrow\quad \boxed{x=-\frac{2}{3}}$$

So $\quad (1+x)^n=\left(1-\frac{2}{3}\right)^{-\frac{1}{2}}=\left(\frac{1}{3}\right)^{-\frac{1}{2}}$

$$=(3)^{\frac{1}{2}}\ =\sqrt{3}$$

This implies

$$1+y=\sqrt{3}$$

On squaring both sides

$$(1+y)^2 = \left(\sqrt{3}\right)^2$$

$$\Rightarrow\ 1+2y+y^2 = 3 \quad \Rightarrow\ 1+2y+y^2-3 = 0$$

$$\Rightarrow\ y^2+2y-2=0 \quad \text{Proved.}$$

Question # 12

If $2y = \frac{1}{2^2} + \frac{1\cdot 3}{2!}\cdot\frac{1}{2^4} + \frac{1\cdot 3\cdot 5}{3!}\cdot\frac{1}{2^6} + \ldots$, then prove that $4y^2+4y-1=0$.

Solution $\quad 2y = \frac{1}{2^2} + \frac{1\cdot 3}{2!}\cdot\frac{1}{2^4} + \frac{1\cdot 3\cdot 5}{3!}\cdot\frac{1}{2^6} + \ldots$

Adding 1 on both sides

$$1+2y = 1+\frac{1}{2^2} + \frac{1\cdot 3}{2!}\cdot\frac{1}{2^4} + \frac{1\cdot 3\cdot 5}{3!}\cdot\frac{1}{2^6} + \ldots$$

Let the given series be identical with

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots$$

This implies

$$nx = \frac{1}{2^2} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ \text{(i)}$$

$$\frac{n(n-1)}{2!}x^2 = \frac{1\cdot 3}{2!}\cdot\frac{1}{2^4} \ \ldots\ldots\ldots\ \text{(ii)}$$

From (i) $\quad nx = \frac{1}{4} \quad \Rightarrow\ x = \frac{1}{4n} \ \ldots\ldots\ldots\ldots\ \text{(iii)}$

Putting value of x in (ii)

$$\frac{n(n-1)}{2!}\left(\frac{1}{4n}\right)^2 = \frac{1\cdot 3}{2!}\cdot\frac{1}{2^4}$$

$$\Rightarrow\ \frac{n(n-1)}{2}\left(\frac{1}{16n^2}\right) = \frac{3}{2}\cdot\frac{1}{16}$$

$$\Rightarrow\ \frac{n-1}{n} = 3 \quad \Rightarrow\ n-1 = 3n$$

$$\Rightarrow\ n-3n = 1 \quad \Rightarrow\ -2n = 1 \quad \Rightarrow\ \boxed{n = -\frac{1}{2}}$$

Putting value of n in equation (iii)

$$x = \frac{1}{4\left(-\frac{1}{2}\right)} \ \Rightarrow\ \boxed{x = -\frac{1}{2}}$$

So

$$(1+x)^n = \left(1-\frac{1}{2}\right)^{-\frac{1}{2}}$$

$$= \left(\frac{1}{2}\right)^{-\frac{1}{2}} = (2)^{\frac{1}{2}} = \sqrt{2}$$

This implies
$$1+2y=\sqrt{2}$$
On squaring both sides
$$\left(1+2y\right)^2=\left(\sqrt{2}\right)^2$$
$$\Rightarrow\ 1+4y+4y^2=4 \qquad \Rightarrow\ 1+4y+4y^2-2=0$$
$$\Rightarrow\ 4y^2+4y-1=0 \qquad \text{Proved}$$

Question # 13

If $y=\frac{2}{5}+\frac{1\cdot 3}{2!}\left(\frac{2}{5}\right)^2+\frac{1\cdot 3\cdot 5}{3!}\left(\frac{2}{5}\right)^3+\ldots$, then prove that $y^2+2y-4=0$.

Solution $\quad y=\frac{2}{5}+\frac{1\cdot 3}{2!}\left(\frac{2}{5}\right)^2+\frac{1\cdot 3\cdot 5}{3!}\left(\frac{2}{5}\right)^3+\ldots$

Adding 1 on both sides
$$1+y=1+\frac{2}{5}+\frac{1\cdot 3}{2!}\left(\frac{2}{5}\right)^2+\frac{1\cdot 3\cdot 5}{3!}\left(\frac{2}{5}\right)^3+\ldots$$
Let the given series be identical with
$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\ldots$$
This implies
$$nx=\frac{2}{5}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (\text{i})$$
$$\frac{n(n-1)}{2!}x^2=\frac{1\cdot 3}{2!}\left(\frac{2}{5}\right)^2\ \ldots\ldots\ldots\ldots\ (\text{ii})$$
From (i) $\quad nx=\frac{2}{5}\quad\Rightarrow\ x=\frac{2}{5n}\ \ldots\ldots\ldots\ldots\ (\text{iii})$

Putting value of x in (ii)
$$\frac{n(n-1)}{2!}\left(\frac{2}{5n}\right)^2=\frac{1\cdot 3}{2!}\left(\frac{2}{5}\right)^2$$
$$\Rightarrow\ \frac{n(n-1)}{2}\left(\frac{4}{25n^2}\right)=\frac{3}{2}\left(\frac{4}{25}\right)$$
$$\Rightarrow\ \frac{n-1}{n}=3 \quad\Rightarrow\ n-1=3n \ \Rightarrow\ n-3n=1$$
$$\Rightarrow\ -2n=1 \quad\Rightarrow\ \boxed{n=-\frac{1}{2}}$$
Putting value of n in equation (iii)
$$x=\frac{2}{5\left(-\frac{1}{2}\right)}\ \Rightarrow\ \boxed{x=-\frac{4}{5}}$$

So $(1+x)^n = \left(1-\frac{4}{5}\right)^{-\frac{1}{2}} = \left(\frac{1}{5}\right)^{-\frac{1}{2}}$

$$= (5)^{\frac{1}{2}} = \sqrt{5}$$

This implies

$$1+y=\sqrt{5}$$

On squaring both sides

$$(1+y)^2 = \left(\sqrt{5}\right)^2$$

$$\Rightarrow\ 1+2y+y^2=5 \quad \Rightarrow\ 1+2y+y^2-5=0$$

$$\Rightarrow\ y^2+2y-4=0$$ Proved.

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