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Reduced equivalent form of a financial structure

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ABSTRACT

We consider the two-date model of a financial exchange economy $(\mathcal{E}, \mathcal{F})$, with agents' portfolio restrictions either represented by finitely many linear inequality constraints or satisfying Hart's (1974) Weak No Market Arbitrage condition. The economy $(\mathcal{E}, \mathcal{F})$ is shown to have the same *consumption equilibria* as a *reduced* economy $(\mathcal{E}, \mathcal{F}')$, for which the set of admissible portfolio allocations is bounded. Building upon the equilibrium existence result for reduced financial economies $(\mathcal{E}, \mathcal{F}')$ (Aouani and Cornet, 2009), we then deduce the existence of equilibra of $(\mathcal{E}, \mathcal{F})$, under standard assumptions on the consumption side and under the aforementioned assumption on the financial side.

1. Introduction

Restricted participation to financial markets refers to the fact that agents face constraints on their portfolio holdings. These constraints are usually described by a collection of subsets $Z_i \subset \mathbb{R}^J$ $(i \in \mathbf{I})$, one for each agent $i \in \mathbf{I}$, of the portfolio space $\mathbb{R}^J$. The economic relevance and interest in considering restricted participation dates back to the seminal papers of Radner (1972) where agents face short sales constraints and Siconolfi (1989), Cass (1984, 2006) for general closed convex portfolio sets Z_i. The presence of such portfolio constraints is a natural cause of market incompleteness – even if there exist enough assets to hedge all risks – and allows to capture a wide range of imperfections in the financial markets, such as collateral requirements, margin requirements, "combo" sales, short selling constraints, and other institutional constraints. These constraints can be either exogenously given, or arise endogenously due to regulatory, institutional (fiscal), or budgetary considerations that may depend on market prices and/or commodity purchases; see Cass et al. (2001), Carosi et al. (2009), and Seghir and Torres-Martinez (2011). We refer to Elsinger and Summer (2001) for an extensive discussion of institutional constraints and how to model them in a general financial framework.

The equilibrium existence problem in the context of restricted participation had a renewed interest since the first work by Radner (1972) and Siconolfi (1989). Linear equality constraints are considered by Balasko et al. (1990) in an economy with nominal assets, and by Polemarchakis and Siconolfi (1997) with real assets, whereas Aouani and Cornet (2009) study linear equality and *inequality* constraints with either nominal or numéraire assets. More recently, the "general" case of closed, convex portfolio sets Z_i, as in Siconolfi (1989), is considered by Angeloni and Cornet (2006) and Aouani and Cornet (2009) for real assets, and by Martins-da-Rocha and Triki (2005), Hahn and Won (2007), and Cornet and Gopalan (2010) for nominal assets.

A key step in the proof of existence of a financial equilibrium with nominal, numéraire or real assets is to show that equilibrium portfolios can be, a priori, chosen in a bounded set. This is a

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standard argument in the absence of redundant assets, or equivalently if the payoff matrix has full column rank. With unrestricted participation, there is no loss of generality in making this "Full Rank Assumption;" indeed, one can remove the redundant assets (by deleting the redundant columns of the payoff matrix), keeping the same consumption equilibria. The situation is drastically different with portfolio constraints and there are no a priori grounds for the standard Full Rank Assumption; as emphasized by Balasko et al. (1990), one significant source of restricted participation is financial intermediation which typically involves redundancy.

With portfolio restrictions defined by finitely many linear equality constraints, Balasko et al. (1990) develop a "reduction" procedure to overcome this obstacle, reducing the original financial economy, keeping the same consumption equilibria, so that each agent's new portfolio set is a subspace having the same dimension as the wealth space it generates; this non-redundancy-type condition was introduced by Siconolfi (1989) to show existence of equilibrium. Aouani and Cornet (2009) extend the above procedure to portfolio sets defined by finitely many linear equality and/or inequality constraints; by appropriately modifying agents' portfolio sets, they obtain a new – say *reduced*– financial structure satisfying a non-redundancy-type condition, weaker than the one in Siconolfi (1989), keeping the correspondence between the consumption equilibria. Furthermore, they show the existence of equilibria for reduced financial economies, and then deduce the existence of equilibria of the original economy.

The main purpose of this paper is to go beyond the case of linear constraints and provide an existence result when the financial structure satisfies in particular Hart's (1974) *weak no market arbitrage* condition and in fact a weaker closedness assumption that also encompasses the case of linear constraints. Although existence of equilibrium was the driving force of this work, it becomes an immediate consequence of the conjunction of our main result concerning the existence of reduced equivalent financial structures and the equilibrium existence result for reduced financial economies in Aouani and Cornet (2009). Since simply removing redundant assets would considerably change the nature of the market by altering wealth transfer sets, we propose instead, to remove some of the portfolios that are "useless". This elimination of *useless* portfolios goes beyond the process initiated by Werner (1987) as explained in our companion paper Aouani and Cornet (2008).

The paper is organized as follows. In Section 2, we describe the model of a financial exchange economy and two financial structures $\mathcal{F}$ and $\mathcal{F}'$ are defined to be equivalent when the financial exchange economies $(\mathcal{E},\mathcal{F})$ and $(\mathcal{E},\mathcal{F}')$ have the same consumption equilibria for every "standard" real exchange economy $\mathcal{E}$. In Section 3.1 we state our first result on the existence of a reduced equivalent form of a financial structure (Theorem 1), after defining a reduced financial structure as having a bounded set of admissible portfolio allocations. In Section 3.2, we provide statements and proofs for the equilibrium existence theorems (Theorems 2 and 3) as a direct consequence of the existence of the reduced equivalent form (Theorem 1) and our previous existence result (Aouani and Cornet, 2009) when the financial structure is reduced. Section 3.3 provides sufficient conditions under which Hart's (1974) Weak No Market Arbitrage condition holds. Section 4 is devoted to the proof of our main result (Theorem 1) as a consequence of a sharper result (Theorem 4) under a closedness assumption that encompasses both Weak No Market Arbitrage and linear constraints. In Appendix A we first study the relationships between the reducibility property, Hart's (1974) condition, and the weaker closedness assumption (see Proposition A.1), and then provide proofs for all the lemmas stated previously in the paper.

2. The two-date model of a financial economy

2.1. Standard exchange economies

We[1] consider the basic stochastic model with two dates: $t = 0$ (today) and $t = 1$ (tomorrow). At the second date, there is a nonempty finite set $\mathbf{S}:=\{1,\ldots,S\}$ of states of nature, one (and only one) of which prevails at time $t = 1$ and is only known at time $t = 1$. For convenience, $s = 0$ denotes the state of the world (known with certainty) at date 0 and we let $\bar{\mathbf{S}}:=\{0\}\cup\mathbf{S} = \{0,1,\ldots,S\}$. At each state, today and tomorrow, there is a spot market for a positive number ℓ of perfectly divisible perishable physical goods. A commodity is thus a couple (h, s), specifying the physical good $h = 1,\ldots,\ell$ and the state $s = 0, 1,\ldots,S$ at which it is available. Thus the commodity space is $\mathbb{R}^L$, where $L = \ell(1 + S)$ and we will use the notation $x = (x(s))_{s\in\bar{\mathbf{S}}}\in\mathbb{R}^L$ (resp. $p = (p(s))_{s\in\bar{\mathbf{S}}}$), where $x(s) = (x_1(s),\ldots,x_\ell(s))\in\mathbb{R}^\ell$ (resp. $p(s)$) denotes the spot consumption (resp. price) at node $s\in\bar{\mathbf{S}}$.

There Is a Nonempty finite set $\mathbf{I}:=\{1,\ldots,I\}$ of consumers, each of whom is endowed with a consumption set $X_i\subset\mathbb{R}^L$, a preference correspondence P_i, from $\prod_{k\in\mathbf{I}}X_k$ to X_i, and an endowment vector $e_i\in\mathbb{R}^L$. The set X_i is the set of her possible consumptions, and for $x\in\prod_{i\in\mathbf{I}}X_i$, $P_i(x)$ is the set of consumption plans in X_i which are strictly preferred to x_i by consumer i, given the consumption plans $(x_{i'})_{i'\neq i}$ of the other agents. The exchange economy can thus be summarized by

$$\mathcal{E} = (\mathbf{I},\mathbf{S},(X_i,P_i,e_i)_{i\in\mathbf{I}}).$$

We make the following standard assumptions **C1** –**C6** on the economy $\mathcal{E}$ and we denote by $\mathcal{A}(\mathcal{E})$ the set of attainable allocations of $\mathcal{E}$, that is,

$$\mathcal{A}_{\mathcal{E}} = \left\{(x_i)_{i\in}\in\prod_{i\in\mathbf{I}}X_i : \sum_{i\in\mathbf{I}}x_i = \sum_{i\in\mathbf{I}}e_i\right\}.$$

Consumption Assumption C For every $i\in\mathbf{I}$ and for every $x = (x_k)_{k\in\mathbf{I}}\in\prod_{k\in\mathbf{I}}X_k$

C1 Consumption Sets: X_i is a closed, convex, bounded below subset of $\mathbb{R}^L$;

[1] We shall use hereafter the following notations. If $\mathbf{I}$ is a finite set, whose number of elements is I, the space $\mathbb{R}^I$ (identified to the space $\mathbb{R}^{\mathbf{I}}$ of functions $x:\mathbf{I}\to\mathbb{R}$ whenever necessary) is endowed with the scalar product $x\cdot y:=\sum_{i=1}^I x_iy_i$, and we denote by $\|x\|:=\sqrt{x\cdot x}$ the Euclidean norm, $B_I(x,r):=\{y\in\mathbb{R}^I : \|y-x\|\le r\}$, the closed ball centered at $x\in\mathbb{R}^I$ of radius $r>0$. For $x=(x_i)$ and $y=(y_i)$ in $\mathbb{R}^I$, the notation $x\ge y$ (resp. $x>y, x\gg y$) means that, for every i, $x_i\ge y_i$ (resp. $x\ge y$ and $x\ne y$, resp. $x_i>y_i$) and we let $\mathbb{R}^I_+=\{x\in\mathbb{R}^I : x\ge 0\}$, $\mathbb{R}^I_{++}=\{x\in\mathbb{R}^I : x\gg 0\}$. Let $X\subset\mathbb{R}^I$, the span of X is the linear subspace of $\mathbb{R}^I$, denoted $<X>$, which is the set of all the K-linear combinations $\sum_{k=1}^K\alpha_kx_k$ of vectors $x_k\in X$ for every integer K, and we denote by int X, cl X, respectively, the interior and the closure of X. Consider a $I\times J$-matrix A with I rows and J columns, with entries $A_i^j (i\in I, j\in J)$, we denote by A_i the ith row of A (hence a row vector, i.e., a $(1\times J)$-matrix, often identified to a vector in $\mathbb{R}^J$ when there is no risk of confusion) and A^j denotes the jth column of A (hence a column vector, i.e., a $I\times 1$-matrix, which may similarly be identified to a vector in $\mathbb{R}^I$). If there is no risk of confusion, we will use the same notation for the $I\times J$-matrix A and the associated linear mapping $A:\mathbb{R}^J\to\mathbb{R}^I$. We shall denote by $kerA:=\{x\in\mathbb{R}^J : Ax=0\}$ the kernel of A, by $\mathrm{Im}\,A:=\{Ax : x\in\mathbb{R}^J\}$ the image of A, and by rank A the rank of the matrix A, that is, the dimension of $\mathrm{Im}\,A$. We also denote $kerA$ by $\{A=0\}$ and we let $\{A\ge 0\}:=\{x\in\mathbb{R}^J : Ax\ge 0\}$. When L is a subset of $\mathbb{R}^J$, we define the orthogonal set to L by $L^\perp:=\{z\in\mathbb{R}^J : z\cdot\xi=0 \text{ for all } \xi\in L\}$. When L is a linear space and $\varphi\in\mathbb{R}^J$, we denote by $\mathrm{proj}_L\varphi$ (resp. $\mathrm{proj}_{L^\perp}\varphi$) the orthogonal projection of φ on L (resp. on $L^\perp$), that is, the unique $\alpha\in L$ (resp. $\beta\in L^\perp$) such that $\varphi-\alpha\in L^\perp$ (resp. $\varphi-\beta\in L$).

C2 Continuity: The correspondence P_i, from $\prod_{k\in\mathbf{I}} X_k$ to X_i, is lower semicontinuous[2] with open values in X_i (for the relative topology of X_i);
C3 Convexity: $P_i(x)$ is convex;
C4 Irreflexivity: $x_i \notin P_i(x)$;
C5 Local Non-Satiation LNS: $\forall x \in \mathcal{A}_{\mathcal{E}}$:
(a) $\forall s \in \mathbf{S}$, $\exists x_i'(s) \in \mathbb{R}^\ell$, $(x_i'(s), x_i(-s)) \in P_i(x)$,[3]
(b) $\forall y_i \in P_i(x)$, $(x_i, y_i] \subset P_i(x)$;
C6 Consumption Survival CS: $e_i \in \operatorname{int} X_i$.

We note that these assumptions are standard in a model with nonordered preferences; the assumptions on P_i are satisfied in particular when agents' preferences are represented by utility functions that are continuous, strongly monotonic, and quasi-concave. An exchange economy $\mathcal{E}$ satisfying Assumption **C** will be called *standard*.

2.2. Financial structures

Agents may operate financial transfers across states in $\bar{\mathbf{S}}$ (i.e. across the two dates and across the states of the second date) by exchanging finitely many assets $j \in \mathbf{J} := \{1, \ldots, J\}$. The assets are traded at the first date ($t = 0$) and yield payoffs $V_s^j(p)$ (for a given commodity price $p \in \mathbb{R}^L$) at the second date ($t = 1$), contingent on the realization of the state of nature $s \in \mathbf{S}$. So, the payoff of asset j across tomorrow states is described by the mapping $p \mapsto V^j(p) := (V_s^j(p))_{s\in S} \in \mathbb{R}^S$. The financial structure is described by the payoff matrix mapping $V : p \mapsto V(p)$, where $V(p)$ is the $S \times J$-matrix, whose columns are the payoffs $V^j(p)$ ($j = 1, \ldots, J$) of the J assets. A portfolio $z = (z_j)_{j\in\mathbf{J}} \in \mathbb{R}^J$ specifies the quantities $|z_j|$ ($j \in \mathbf{J}$) of each asset j, with the convention that the asset j is bought if $z_j > 0$ and sold if $z_j < 0$. Thus $V(p)z$ is its random payoff across states at time $t = 1$, and $V_s(p) \cdot z$ is its payoff if state s prevails. Each agent i is endowed with a portfolio set $Z_i \subset \mathbb{R}^J$, which represents the constraints faced by the agent. The financial characteristics, referred to as the financial structure are summarized by

$$\mathcal{F} = (\mathbf{I}, \mathbf{S}, \mathbf{J}, V, (Z_i)_{i\in\mathbf{I}})$$

2.3. Equivalent financial structures

Given commodity and asset prices $(p, q) \in \mathbb{R}^L \times \mathbb{R}^J$, the budget set of consumer i is[4]

$$\begin{aligned} B_i(p,q,\mathcal{E},\mathcal{F}) &= \left\{ (x_i,z_i) \in X_i \times Z_i : \begin{array}{l} p(0)\cdot x_i(0) + q\cdot z_i \le p(0)\cdot e_i(0) \\ p(s)\cdot x_i(s) \le p(s)\cdot e_i(s) + V_s(p)\cdot z_i, \quad \forall s \in \mathbf{S} \end{array} \right\} \\ &= \{(x_i,z_i) \in X_i \times Z_i : \quad p\Box(x_i - e_i) \le W(p,q)z_i\}, \end{aligned}$$

where $W(p, q)$ denotes the total payoff matrix, that is, the $(1 + S) \times J$-matrix $\begin{bmatrix} -q \\ V(p) \end{bmatrix}$. The budget set will be simply denoted $B_i(p, q)$ when there is no risk of confusion. We now introduce the standard equilibrium notion in this model.

Definition 1. An *equilibrium* of the financial exchange economy $(\mathcal{E}, \mathcal{F})$ is a list $(\bar{p}, \bar{x}, \bar{q}, \bar{z}) \in \mathbb{R}^L \times (\mathbb{R}^L)^I \times \mathbb{R}^J \times (\mathbb{R}^J)^I$ such that

(i) for every i, $(\bar{x}_i, \bar{z}_i)$ maximizes the preference P_i in the budget set $B_i(\bar{p}, \bar{q})$, in the sense that

$$(\bar{x}_i, \bar{z}_i) \in B_i(\bar{p}, \bar{q}) \text{ and } B_i(\bar{p}, \bar{q}) \cap (P_i(\bar{x}) \times Z_i) = \varnothing,$$

(ii) [Market Clearing] $\sum_{i\in\mathbf{I}} \bar{x}_i = \sum_{i\in\mathbf{I}} e_i$ and $\sum_{i\in\mathbf{I}} \bar{z}_i = 0$.

A *consumption equilibrium* of $(\mathcal{E}, \mathcal{F})$ is a list $(\bar{p}, \bar{x}) \in \mathbb{R}^L \times (\mathbb{R}^L)^I$ such that there exist $(\bar{q}, \bar{z}) \in \mathbb{R}^J \times (\mathbb{R}^J)^I$ and $(\bar{p}, \bar{x}, \bar{q}, \bar{z})$ is an equilibrium of $(\mathcal{E}, \mathcal{F})$.

We introduce an equivalence relation on the set of all financial structures defined on the same set of agents **I** and the same set of states **S**. The intuition behind this definition is the following. Financial structures allow agents to transfer wealth across states of nature and thereby give them the possibility to enlarge their budget set. Hence if, regardless of the standard exchange economy $\mathcal{E}$, consumption equilibria are the same when agents carry out their financial activities through two different structures $\mathcal{F}$ and $\mathcal{F}'$, then $\mathcal{F}$ and $\mathcal{F}'$ are said to be equivalent.

Definition 2. Consider two financial structures $\mathcal{F} = (\mathbf{I}, \mathbf{S}, \mathbf{J}, V, (Z_i)_i)$ and $\mathcal{F}' = (\mathbf{I}, \mathbf{S}, \mathbf{J}', V', (Z_i')_i)$. Then $\mathcal{F}$ and $\mathcal{F}'$ are said to be equivalent, denoted $\mathcal{F} \sim \mathcal{F}'$, if for every standard exchange economy $\mathcal{E}$, the financial exchange economies $(\mathcal{E}, \mathcal{F})$ and $(\mathcal{E}, \mathcal{F}')$ have the same consumption equilibria.

We now recall that equilibrium asset prices preclude unbounded arbitrage opportunities under Local Non-Satiation (**LNS**). We denote by $\boldsymbol{A}Z$ the asymptotic cone[5] of a nonempty set $Z \subset \mathbb{R}^J$.

Proposition 1. *Assume* **LNS** *and that the portfolio sets Z_i ($i \in \mathbf{I}$) are closed and convex. If $(\bar{p}, \bar{x}, \bar{q}, \bar{z})$ is an equilibrium of the economy $(\mathcal{E}, \mathcal{F})$, then $\bar{q}$ is arbitrage-free at $\bar{p}$, in the sense that there does not exist a consumer i and $\zeta_i \in \boldsymbol{A}Z_i$ such that $W(\bar{p}, \bar{q})\zeta_i > 0$, that is*

$$W(\bar{p}, \bar{q}) \left(\bigcup_i \boldsymbol{A}Z_i \right) \cap \mathbb{R}_+^{\bar{\mathbf{S}}} = \{0\}.$$

We denote by $\mathbf{Q}_{\mathcal{F}}(p)$ the set of arbitrage-free asset prices at $p \in \mathbb{R}^L$.

Given the financial structure $\mathcal{F} = (V, (Z_i)_{i\in\mathbf{I}})$, we denote $\boldsymbol{Z}_{\mathcal{F}} := < \cup_{i\in\mathbf{I}} Z_i >$ the vector space spanned by the portfolio sets Z_i ($i \in \mathbf{I}$).

Remark 1. $\boldsymbol{Z}_{\mathcal{F}}$ is the space where financial activity takes place. As a consequence, in the following, we will mainly consider asset prices in the set $\mathbf{Q}_{\mathcal{F}}(p) \cap \boldsymbol{Z}_{\mathcal{F}}$, which are the only ones that "matter". More precisely, if $\bar{q}$ is an equilibrium asset price (resp. arbitrage-free asset price), then $\mathrm{proj}_{\boldsymbol{Z}_{\mathcal{F}}} \bar{q}$ is also an equilibrium asset price (resp. arbitrage-free asset price); notice that $\bar{q} \cdot z_i = \mathrm{proj}_{\boldsymbol{Z}_{\mathcal{F}}} \bar{q} \cdot z_i$ for every $i \in \mathbf{I}$ and for every $z_i \in Z_i$.

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[2] Let Φ be a correspondence from X to Y, that is, Φ is a mapping from X to 2^Y. Then Φ is said to be lower semicontinuous (l.s.c.) at $x_o \in X$, if for every open set $V \subset Y$ such that $\Phi(x_o) \cap V \neq \varnothing$, there exists an open neighborhood U of x_o in X such that $\Phi(x) \cap V \neq \varnothing$ for every $x \in U$. The correspondence Φ is said to be l.s.c. if it is l.s.c. at every point of X. Finally, we denote by $G(\Phi) := \{(x, y) \in X \times Y : y \in \Phi(x)\}$ the graph of Φ.

[3] Given $x_i = (x_i(s))_{s\in\bar{\mathbf{S}}} \in \mathbb{R}^{\ell\bar{\mathbf{S}}}$ and $s \in \bar{\mathbf{S}}$, we let $x_i(-s) := (x_i(s'))_{s'\neq s}$ and without any risk of confusion we will write $x_i = (x_i(s), x_i(-s))$.

[4] For every $p = (p(s))_{s\in\bar{\mathbf{S}}}$, $x = (x(s))_{s\in\bar{\mathbf{S}}}$ in $\mathbb{R}^L$, we denote by $p\Box x$ the vector $(p(s)\cdot x(s))_{s\in\bar{\mathbf{S}}}$.

[5] The asymptotic cone of a nonempty subset Z of $\mathbb{R}^J$ is the set $\boldsymbol{A}Z := \{\lim_n \lambda^n z^n : (\lambda^n)_n \downarrow 0 \text{ and } z^n \in Z \text{ for all } n\}$. As a consequence from the definition, one has $\boldsymbol{A}(\mathrm{cl}Z) = \boldsymbol{A}Z$ and we refer to Debreu (1959) for a general reference. When Z is additionally assumed to be convex, then $\boldsymbol{A}Z = 0^+(\mathrm{cl}Z)$, where $0^+(C) := \{\zeta \in \mathbb{R}^J : \zeta + C \subset C\}$ is the recession cone of the convex set $C \subset \mathbb{R}^J$ (see Rockafellar, 1970). When Z is convex, the inclusion $0^+(Z) \subset \boldsymbol{A}Z$ holds but may be strict when Z is not closed. When Z is convex and $0 \in Z$ we will use extensively the property that $\boldsymbol{A}Z \subset \mathrm{cl}Z$.

3. Main results

3.1. Equivalent reduced form of a financial structure

We make the following assumptions on the financial side of the economy.

F0: The set $\boldsymbol{A}_{\mathcal{F}}(p):=\boldsymbol{A}\left(\sum_{i\in\mathbf{I}}(Z_i\cap\{V(p)\geq 0\})\right)$ is independent of p (hence denoted $\boldsymbol{A}_{\mathcal{F}}$ hereafter).

F1: For every $i\in\mathbf{I}$, Z_i is closed, convex, contains 0, and $V:\mathbb{R}^L\to\mathbb{R}^{S\times J}$ is continuous.

F2: One of the following two conditions holds:

(i) For all $i\in\mathbf{I}$, $Z_i=P_i+K_i$ where P_i is polyhedral convex[6] and K_i is compact and convex.

(ii) Weak No Market Arbitrage (**WNMA**) [Hart (1974)]: for all $p\in\mathbb{R}^L$, $(\forall i\in\mathbf{I},\ \zeta_i\in\boldsymbol{A}Z_i\cap kerV(p)$ and $\sum_{i\in\mathbf{I}}\zeta_i=0)\Rightarrow \forall i\in\mathbf{I},\ \zeta_i\in-\boldsymbol{A}Z_i$.

FS [Financial Survival]: $\forall i\in\mathbf{I}, \forall p\in\mathbb{R}^L, p(0)=0, \forall q\in \mathrm{cl}[\mathbf{Q}_{\mathcal{F}}(p)\cap\boldsymbol{Z}_{\mathcal{F}}]$ [7], $q\neq 0$, $\exists\,\zeta_i\in Z_i$, $q\cdot\zeta_i<0$.

We say that the financial structure $\mathcal{F}$ is *standard* if it satisfies both Assumptions **F0** and **F1**.

Assumption **F0** allows to cover the cases of financial structures with nominal assets (since the payoff matrix V does not depend on the price p) and numéraire assets (up to a modification of the payoff matrix); see next Section 3.2. However, **F0** does not cover the general case of real assets when the rank of $V(p)$ may change. In that case Hart's (1975) counter-example applies and existence can be shown only generically in the unrestricted case; we refer to Duffie and Shafer (1985, 1986) and the extensive body of literature that built upon their argument, see e.g. Geanakopolos and Shafer (1990), Hirsch et al. (1990), Husseini et al. (1990), and Bich and Cornet (2004, 2009).

Assumption **F1** is the general framework of the paper and needs no further comments. In Section 3.3, Propositions 2 and 3 provide sufficient conditions for Assumption **F2** to hold true; the convex polyhedral framework was considered in Aouani and Cornet (2009) to generalize the linear *equality* constraints' framework of Balasko et al. (1990), and Hart's (1974) Weak No Market Arbitrage is standard in the literature on unbounded arbitrage.

Assumption **FS** is a financial survival assumption through asset markets; it ensures that every agent is able to borrow through the financial markets, that is, transfer "money" from tomorrow to today. Note that Angeloni and Cornet (2006) assume a different financial survival assumption, not directly comparable to **FS**, and do not assume **F0**.

Definition 3. The financial structure $\mathcal{F}=(\mathbf{I},\mathbf{S},\mathbf{J},V,(Z_i)_i)$ is said to be reduced if for all $p\in\mathbb{R}^L$, all $v=(v_i)_i\in(\mathbb{R}^S)^I$, the set $\mathcal{A}_{\mathcal{F}}(p,v)$ of admissible portfolio allocations is bounded, where

$$\mathcal{A}_{\mathcal{F}}(p,v):=\left\{(z_i)_i\in\prod_i Z_i:\forall i,\ V(p)z_i\geq v_i,\ \sum_{i\in\mathbf{I}}z_i=0\right\}.$$

The definition of $\mathcal{A}_{\mathcal{F}}(p,v)$ contains the constraints "$V(p)z_i\geq v_i$" since equilibrium portfolio allocations satisfy these constraints for some $\underline{v}_i\in\mathbb{R}^S(i\in\mathbf{I})$ small enough; indeed from agents' budget constraints, at equilibrium one has $\bar{p}(s)\cdot(\bar{x}_i(s)-e_i(s))\leq V_s(\bar{p})\cdot\bar{z}_i\ (s\in\mathbf{S})$, and the left-hand side is bounded below when the equilibrium commodity price $\bar{p}$ belongs to $B_L(0,1)$ and equilibrium consumption allocations are bounded (by Assumption **C**).

We point out that every reduced financial structure satisfies **WNMA** (see Proposition A.1). We can now state our main result.

Theorem 1. *Let $\mathcal{F}=(\mathbf{I},\mathbf{S},\mathbf{J},V,(Z_i)_i)$ be a standard (i.e., satisfies **F0** and **F1**) financial structure satisfying **F2**. Then there exists a standard **reduced** financial structure $\mathcal{F}'$ such that:*

(a) *for every standard exchange economy $\mathcal{E}$, every consumption equilibrium of $(\mathcal{E},\mathcal{F}')$ is a consumption equilibrium of $(\mathcal{E},\mathcal{F})$,*

(b) *$\mathcal{F}'$ satisfies the Financial Survival Assumption **FS** if $\mathcal{F}$ satisfies **FS**.*

The proof of Theorem 1 is given in Section 4 as a consequence of a sharper result (Theorem 4) that will also give a sufficient condition under which $\mathcal{F}$ and $\mathcal{F}'$ are equivalent; only Assertion (a) is needed in the next section to deduce the existence of equilibria of $(\mathcal{E},\mathcal{F})$.

3.2. Existence of equilibria

We can now state our first result on the existence of equilibria of the economy $(\mathcal{E},\mathcal{F})$.

Theorem 2. *The economy $(\mathcal{E},\mathcal{F})$ admits an equilibrium $(\bar{p},\bar{x},\bar{q},\bar{z})$ such that $\|\bar{p}(0)\|+\|\bar{q}\|=1$ and $\|\bar{p}(s)\|=1$ for $s\in\mathbf{S}$ if it satisfies Assumptions **C**, **F0**, **F1**, **F2**, and **FS**.*

It is worth noting that, under the assumptions of Theorem 2, the set $\mathcal{A}_{\mathcal{E}}$, of admissible consumption allocations, is compact but the set, $\mathcal{A}_{\mathcal{F}}(p,v)$, of admissible portfolio allocations may not be bounded (it is clearly closed). In order to circumvent this difficulty, the proof of Theorem 2 consists in two steps. The first step relies on Theorem 1 which associates an "equivalent"[8] reduced form $\mathcal{F}'$ to the financial structure $\mathcal{F}$. The second step of the proof consists in getting the existence of an equilibrium $(\mathcal{E},\mathcal{F}')$ (using the compactness of the set $\mathcal{A}_{\mathcal{F}'}(p,v)$ since $\mathcal{F}'$ is reduced) as a consequence of Aouani and Cornet (2009) (Theorem 2, p. 777), in fact a slight generalization of it, which appears in the companion Working Paper (where $\mathcal{F}'$ reduced replaces the stronger Condition $\boldsymbol{A}_{\mathcal{F}}\cap-\boldsymbol{A}_{\mathcal{F}}=\{0\}$; see Lemma 2).

Proof of Theorem 2. First, from Theorem 1 we obtain a standard reduced financial structure $\mathcal{F}'$ satisfying **FS** (since $\mathcal{F}$ satisfies **FS**). Second, by the above mentioned existence result, the reduced economy $(\mathcal{E},\mathcal{F}')$ admits an equilibrium $(p',\bar{x},q'',z')$ such that $\|p'(s)\|=1$ for all $s\in\mathbf{S}$. The consumption equilibrium $(p',\bar{x})$ of $(\mathcal{E},\mathcal{F}')$ is a consumption equilibrium of $(\mathcal{E},\mathcal{F})$ by Theorem 1. Hence there exists $(q',\bar{z})$ such that $(p',\bar{x},q',\bar{z})$ is an equilibrium of $(\mathcal{E},\mathcal{F})$. The end of the proof consists in modifying the prices p', q' into $\bar{p}$, $\bar{q}$ satisfying $\|\bar{p}(0)\|+\|\bar{q}\|=1$ and $\|\bar{p}(s)\|=1$ for $s\in\mathbf{S}$ so that $(\bar{p},\bar{x},\bar{q},\bar{z})$ is also an equilibrium of $(\mathcal{E},\mathcal{F})$.[9] □

6 We say that $Z\subset\mathbb{R}^J$ is a polyhedral convex set if it can be defined by finitely many linear inequalities, i.e., $Z:=\{z\in\mathbb{R}^J: Bz\geq b\}$ for some $K\times J$-matrix B and some $b\in\mathbb{R}^K$.

7 In Aouani and Cornet (2009), the financial survival assumption is made for prices $q\in[\mathrm{cl}\mathbf{Q}_{\mathcal{F}}(p)]\cap\boldsymbol{Z}_{\mathcal{F}}$, and we notice that the two conditions are actually equivalent since $\mathrm{cl}[\mathbf{Q}_{\mathcal{F}}(p)\cap\boldsymbol{Z}_{\mathcal{F}}]=[\mathrm{cl}\mathbf{Q}_{\mathcal{F}}(p)]\cap\boldsymbol{Z}_{\mathcal{F}}$. Note that the inclusion $\mathrm{cl}[\mathbf{Q}_{\mathcal{F}}(p)\cap\boldsymbol{Z}_{\mathcal{F}}]\subset[\mathrm{cl}\mathbf{Q}_{\mathcal{F}}(p)]\cap\boldsymbol{Z}_{\mathcal{F}}$ is immediate and we now show the converse inclusion. Let $q\in[\mathrm{cl}\mathbf{Q}_{\mathcal{F}}(p)]\cap\boldsymbol{Z}_{\mathcal{F}}$. Then $q=\lim_n q^n$ for some sequence $(q^n)_n\subset\mathbf{Q}_{\mathcal{F}}(p)$. Since $q\in\boldsymbol{Z}_{\mathcal{F}}$, one has $q=\mathrm{proj}_{\boldsymbol{Z}_{\mathcal{F}}}q=\lim_n\mathrm{proj}_{\boldsymbol{Z}_{\mathcal{F}}}q^n$. Clearly, $(\mathrm{proj}_{\boldsymbol{Z}_{\mathcal{F}}}q^n)_n\subset\boldsymbol{Z}_{\mathcal{F}}$ and $\mathrm{proj}_{\boldsymbol{Z}_{\mathcal{F}}}q^n=q^n+(\mathrm{proj}_{\boldsymbol{Z}_{\mathcal{F}}}q^n-q^n)\in\mathbf{Q}_{\mathcal{F}}(p)+(\boldsymbol{Z}_{\mathcal{F}})^{\perp}\subset\mathbf{Q}_{\mathcal{F}}(p)$ (from the definition of $\mathbf{Q}_{\mathcal{F}}(p)$). Hence $q=\lim_n\mathrm{proj}_{\boldsymbol{Z}_{\mathcal{F}}}q^n\in\mathrm{cl}[\mathbf{Q}_{\mathcal{F}}(p)\cap\boldsymbol{Z}_{\mathcal{F}}]$.

8 For the existence problem (the proof of Theorem 2), we only need the property that every consumption equilibrium of $(\mathcal{E},\mathcal{F}')$ is a consumption equilibrium of $(\mathcal{E},\mathcal{F})$ as stated in Theorem 1. We postpone the proof of the equivalence between $\mathcal{F}$ and its reduced form $\mathcal{F}'$ to Section 4.1, see Theorem 4, a sharper version of Theorem 1.

9 Let $\bar{p}=(\lambda p'(0),(p'(s))_{s\in\mathbf{S}})$ and $\bar{q}=\lambda q'$ with $\lambda=1/(\|p'(0)\|+\|q'\|)$ (so that $\|\bar{p}(0)\|+\|\bar{q}\|=1$). ¿From Local Non-Satiation **LNS** (at $s=0$), we deduce that $\|p'(0)\|+\|q'\|>0$, thus $\lambda=1/(\|p'(0)\|+\|q'\|)$ is well defined; moreover $((\lambda p'(0),(p'(s))_{s\in\mathbf{S}}),\bar{x},\lambda q',\bar{z})$ is an equilibrium of $(\mathcal{E},\mathcal{F})$ (from the positive homogeneity property in $(p(0),q)$ of the budget constraint at $t=0$).

We refer to Aouani et al. (2011) for a counter-example on the non-existence of equilibria in the absence of either Assumption **F2**, or the weaker Closedness Assumption made in Theorem 4.

Remark 2. It is worth mentioning that neither Theorem 1 nor Theorem 2 assumes that there is no redundant asset or equivalently that rank $V(p) = J$ for every p. This assumption is standard in the unconstrained case ($Z_i = \mathbb{R}^J$ for all i) since one can "reduce" the financial structure V by eliminating the redundant assets and associate a new financial structure V' which is "equivalent" to the previous one in the sense of Definition 2. However this process (of eliminating redundant assets) is no longer possible in the constrained case but Theorem 1 provides an alternative way that generalizes the approach of Balasko et al. (1990) and Aouani and Cornet (2009) from the case of linear constraints to the case of general convex sets.

We now turn to the case of nominal and numéraire financial structures. If the financial structure $\mathcal{F}$ is *nominal*, the matrix $V(p)$ does not depend on the commodity price p and is denoted R. A *numéraire asset* is defined as follows. Let us choose a commodity bundle $\nu \in \mathbb{R}^\ell$; a typical example is $\nu = (0, \ldots, 0, 1)$, when theℓth good is chosen as numéraire. A numéraire asset j is a real asset which delivers $R_s^j (\in \mathbb{R})$ units of the bundle ν, i.e., the commodity bundle $R_s^j \nu (\in \mathbb{R}^\ell)$ at state s if it prevails at $t = 1$; thus, given the commodity price $p = (p(s)) \in \mathbb{R}^L$, the payoff at state s is $(V_\nu^j)_s(p) = (p(s) \cdot \nu) R_s^j$. For a numéraire financial structure, i.e., all the assets $j = 1, \ldots, J$ are numéraire assets (for the same commodity bundle ν), we denote R the $S \times J$-matrix with entries R_s^j and, for $p \in \mathbb{R}^L$, we denote $V_\nu(p)$ the associated $S \times J$-payoff-matrix, with entries $(V_\nu^j)_s(p)$.

In the *nominal* case, the set $\mathbf{Q}_{\mathcal{F}}(p)$ of arbitrage-free prices q, i.e., such that

$$\begin{bmatrix} -q \\ R \end{bmatrix} \left(\bigcup_i \boldsymbol{A} Z_i \right) \cap \mathbb{R}_+^{\bar{\mathbf{S}}} = \{0\} \tag{3.1}$$

does not depend on the price p, hence is simply denoted $\mathbf{Q}_R$.

In the *numéraire* case, under the Desirability Assumption (made in **FN0(ii)**) below, if $(\bar{p}, \bar{x}, \bar{q}, \bar{z})$ is an equilibrium, then $\bar{p}(s) \cdot \nu > 0$ for all $s \in \mathbf{S}$ (see the proof of Lemma 2 in Aouani and Cornet, 2009), hence $\mathbf{Q}_{\mathcal{F}}(\bar{p}) = \mathbf{Q}_R$ as defined above by (3.1).

Thus, in both nominal and numéraire cases, every equilibrium asset price $\bar{q}$ belongs to $\mathbf{Q}_R$ (by Proposition 1); our second existence result will modify the Financial Survival Assumption accordingly. We now state the following general assumptions on the financial side; see Aouani and Cornet (2009) for a thorough discussion.

FN0: The financial structure $\mathcal{F}$ is either **(i) nominal**, i.e., $V(p) = R$ is independent of p, **or (ii) numéraire**, i.e., $V(p) = V_\nu(p)$ for some $\nu \in \mathbb{R}^\ell$, for every agent i the correspondence P_i has an open graph, and the commodity bundle $\nu \in \mathbb{R}^\ell$ is desirable at every state $s \in \mathbf{S}$, i.e., for all $x \in \mathcal{A}(\mathcal{E})$, for all $t > 0$, $(x_i(s) + t\nu, x_i(-s)) \in P_i(x)$.

Theorem 3. *The economy $(\mathcal{E}, \mathcal{F})$ has an equilibrium $(\bar{p}, \bar{x}, \bar{q}, \bar{z})$ such that $\|\bar{p}(0)\| + \|\bar{q}\| = 1$ and $\|\bar{p}(s)\| = 1$ for all $s \in \mathbf{S}$ if it satisfies Assumptions* **C**, **FN0**, **F1**, **F2**, *together with*

FNS: $\forall i \in \mathbf{I}$, $\forall q \in \mathrm{cl}[\mathbf{Q}_R \cap Z_{\mathcal{F}}]$, $q \neq 0$, $\exists \zeta_i \in Z_i$, $q \cdot \zeta_i < 0$.

Proof. In the *nominal* case, the proof of Theorem 3 is a straightforward consequence of Theorem 2. Consider now a financial economy $(\mathcal{E}, \mathcal{F})$ with *numéraire* assets, we define $\mathcal{F}^\varepsilon = (V^\varepsilon, (Z_i)_i)$ for $\varepsilon > 0$, by taking the same portfolio sets Z_i as for $\mathcal{F}$ and we let

$$V^\varepsilon(p) = \begin{bmatrix} \max\{\varepsilon, p(1) \cdot \nu\} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \max\{\varepsilon, p(S) \cdot \nu\} \end{bmatrix} R.$$

First, the financial structure $\mathcal{F}^\varepsilon$ satisfies **F0**, **F1**, **F2**, and **FS**. Indeed, $\{V^\varepsilon(p) \geq 0\} = \{R \geq 0\}$ for every $p \in \mathbb{R}^L$, hence $\mathcal{F}^\varepsilon$ satisfies **F0**. Assumptions **F1** and **F2** are obviously satisfied by $\mathcal{F}^\varepsilon$ since $\mathcal{F}$ satisfies **F1**, **F2**, and $\mathcal{F}^\varepsilon$ satisfies **FS** since $\mathcal{F}$ satisfies **FNS** and $\mathbf{Q}_{\mathcal{F}^\varepsilon}(p) = \mathbf{Q}_R$ for every p.

Second, for $\varepsilon > 0$ small enough, every equilibrium $(\bar{p}, \bar{x}, \bar{q}, \bar{z})$ of $(\mathcal{E}, \mathcal{F}^\varepsilon)$ such that $\|\bar{p}(s)\| = 1$ for $s \in \mathbf{S}$ is an equilibrium of the economy $(\mathcal{E}, \mathcal{F})$ as shown by Aouani and Cornet (2009).

Consequently, there exists an equilibrium of $(\mathcal{E}, \mathcal{F}^\varepsilon)$, by Theorem 2 and Step 1, and it is also an equilibrium of $(\mathcal{E}, \mathcal{F})$, for $\varepsilon > 0$ small enough, by Step 2. □

We can now state some consequences to Theorem 3. The following corollary extends to the case of consumers with non-ordered preferences the existence results of Cass (1984), Duffie (1987), Werner (1985), and Siconolfi (1989) in the nominal case, Geanakoplos and Polemarchakis (1986) in the numéraire case, and Radner (1972) in the general case of real assets.

Corollary 1. *The economy $(\mathcal{E}, \mathcal{F})$ admits an equilibrium under Assumption* **C**, **F1** *in each of the following cases:*

- **FN0**, **F2**, *and* $0 \in \mathrm{int}\, Z_i$, *for all i.*
- (Cass, 1984; Duffie, 1987; Werner, 1985) *$\mathcal{F}$ consists of nominal assets and $Z_i = \mathbb{R}^J$, $(i \in \mathbf{I})$.*
- (Geanakoplos and Polemarchakis, 1986) *$\mathcal{F}$ consists of numéraire assets, satisfies* **FN0(ii)**, *and $Z_i = \mathbb{R}^J$, $(i \in \mathbf{I})$.*
- (Radner, 1972) *$\mathcal{F}$ satisfies* **F0** *and $Z_i = \{z_i \in \mathbb{R}^J : \|z_i\| \leq r_i\}$, for some $r_i > 0 (i \in \mathbf{I})$.*
- (Radner, 1972) *$\mathcal{F}$ satisfies* **F0** *and $Z_i = \underline{z}_i + \mathbb{R}_+^J$, for some $\underline{z}_i \in -\mathbb{R}_{++}^J (i \in \mathbf{I})$.*
- (Siconolfi, 1989) *$\mathcal{F}$ consists of nominal assets,* **FNS** *holds and $\boldsymbol{A} Z_i \cap ker R = \{0\} (i \in \mathbf{I})$.*

3.3. Portfolio constraints satisfying Assumption F2

As shown by the following Propositions 2 and 3, Assumption **F2** holds in many situations. First, **F2** is satisfied when restrictions on portfolios are defined by finitely many linear inequalities (Aouani and Cornet, 2009); in particular, **F2** is fulfilled when portfolio sets are linear subspaces (Balasko et al., 1990), when portfolio sets are unconstrained, or when there is an exogenous bound on portfolio short sales (Radner, 1972).

Second, Assumption **F2** holds true under **WNMA** (Hart, 1974) that is when, for all p, the family $\{\boldsymbol{A} Z_i \cap ker V(p), i \in \mathbf{I}\}$ is weakly positively semi-independent.[10] In particular, Assumption **F2** holds true under the No Unbounded Arbitrage (**NUBA**) condition (Page, 1987) that is when, for all p, the family $\{\boldsymbol{A} Z_i \cap ker V(p), i \in \mathbf{I}\}$ is positively semi-independent, under Siconolfi (1989)'s assumption ($\boldsymbol{A} Z_i \cap ker V(p) = \{0\}$ for all $i \in \mathbf{I}$), when portfolio sets are bounded, or when there are no redundant assets i.e. rank $V = J$.

[10] A finite family $\{C_i : i \in \mathbf{I}\}$ of nonempty convex cones of $\mathbb{R}^n$ is positively semi-independent (resp. weakly positively semi-independent) if $c_i \in C_i$ for all $i \in \mathbf{I}$ and $\sum_{i \in \mathbf{I}} c_i = 0$ imply that, for all $i \in \mathbf{I}$, $c_i = 0$ (resp. $c_i \in C_i \cap -C_i$).

Proposition 2. *Assumption* **F2** (i) *holds true under each of the following condition.*

(a) *For all* $i \in \mathbf{I}$, $Z_i = \mathbb{R}^J$ *(unconstrained portfolios).*
(b) *For all* $i \in \mathbf{I}$, Z_i *is a linear subspace (linear equality constraints).*
(c) *For all* $i \in \mathbf{I}$, $Z_i = \underline{z}_i + \mathbb{R}^J_+$, *for some* $\underline{z}_i \in -\mathbb{R}^J_+$ *(exogenous bounds on short sales).*
(d) *For all* $i \in \mathbf{I}$, Z_i *is polyhedral convex (linear inequality constraints).*
(e) *For all* $i \in \mathbf{I}$, $Z_i = B_J(0,1)$ *(bounded portfolio sets).*
(f) *For all* $i \in \mathbf{I}$, $Z_i = P_i + K_i$ *where* P_i *is polyhedral convex, and* K_i *is nonempty compact.*

Proposition 3. *Assumption* **F2** (ii) *holds true under each of the following conditions.*

(g) *For all* $p \in \mathbb{R}^L$, $\operatorname{rank} V(p) = J$, *or equivalently,* $\ker V(p) = \{0\}$ *(No redundant assets).*
(h) *For all* $p \in \mathbb{R}^L$ *and for all* $i \in \mathbf{I}$, $\boldsymbol{A}Z_i \cap \ker V(p) = \{0\}$.
(i1) *For all* $p \in \mathbb{R}^L$, $\boldsymbol{A}\left(\sum_{i\in\mathbf{I}} Z_i \cap \{V(p) \geq 0\}\right) \cap -\boldsymbol{A}\left(\sum_{i\in\mathbf{I}} Z_i \cap \{V(p) \geq 0\}\right) = \{0\}$.
(i2) *For all* $p \in \mathbb{R}^L$, $\boldsymbol{A}\left(\sum_{i\in\mathbf{I}} Z_i \cap \ker V(p)\right) \cap -\boldsymbol{A}\left(\sum_{i\in\mathbf{I}} Z_i \cap \ker V(p)\right) = \{0\}$.
(i3) *For all* $p \in \mathbb{R}^L$, $\left(\sum_{i\in\mathbf{I}} \boldsymbol{A}Z_i \cap \{V(p) \geq 0\}\right) \cap -\left(\sum_{i\in\mathbf{I}} \boldsymbol{A}Z_i \cap \{V(p) \geq 0\}\right) = \{0\}$.
(i4) *For all* $p \in \mathbb{R}^L$, $\left(\sum_{i\in\mathbf{I}} \boldsymbol{A}Z_i \cap \ker V(p)\right) \cap -\left(\sum_{i\in\mathbf{I}} \boldsymbol{A}Z_i \cap \ker V(p)\right) = \{0\}$.
(j1) *For all* $p \in \mathbb{R}^L$, *the family* $\left\{\boldsymbol{A}Z_i \cap \{V(p) \geq 0\} : i \in \mathbf{I}\right\}$ *is positively semi-independent.*
(j2) *For all* $p \in \mathbb{R}^L$, *the family* $\left\{\boldsymbol{A}Z_i \cap \ker V(p) : i \in \mathbf{I}\right\}$ *is positively semi-independent.*
(k) *For all* $p \in \mathbb{R}^L$, *the family* $\left\{\boldsymbol{A}Z_i \cap \{V(p) \geq 0\} : i \in \mathbf{I}\right\}$ *is weakly positively semi-independent.*

The proof of Propositions 2 and 3 are left to the reader.

4. Proof of Theorem 1

4.1. A sharper result

We introduce the following Closedness Assumption
Closedness: $\mathcal{F}$ is said to be closed if, for all $p \in \mathbb{R}^L$, the set

$$\mathcal{G}_{\mathcal{F}}(p) := \left\{ \left(V(p)z_1, \ldots, V(p)z_I, \sum_{i\in\mathbf{I}} z_i \right) \in (\mathbb{R}^S)^I \times \mathbb{R}^J : \forall i \in \mathbf{I},\ z_i \in Z_i \right\}$$

is closed.

It is worth pointing out that, for a financial structure $\mathcal{F}$, being reduced implies Assumption **F2**, which implies the above Closedness Assumption; see Proposition A.1 in Appendix A.

Let $\mathcal{F} = (V, (Z_i)_i)$ be a standard and closed financial structure. We consider the financial structure $\mathcal{F}_\pi$ which has the same payoff matrix as $\mathcal{F}$ and the portfolio sets $\mathrm{cl}\pi Z_i$ $(i \in \mathbf{I})$ where π is the orthogonal projection mapping of $\mathbb{R}^J$ on the orthogonal space $(\boldsymbol{L}_{\mathcal{F}})^\perp$ to $\boldsymbol{L}_{\mathcal{F}} := \boldsymbol{A}_{\mathcal{F}} \cap -\boldsymbol{A}_{\mathcal{F}}$. The definition of $\mathcal{F}_\pi = (V, (\mathrm{cl}\pi Z_i)_i)$ can be summarized by

$$\mathcal{F}_\pi = (V, (\mathrm{cl}\pi Z_i)_i), \quad \text{where} \quad \boldsymbol{Z}_{\mathcal{F}} := \langle \cup_i Z_i \rangle,\ \boldsymbol{Z}_{\mathcal{F}_\pi} := \langle \cup_i \mathrm{cl}\pi Z_i \rangle \quad \text{and}$$
$$\boldsymbol{A}_{\mathcal{F}} := \boldsymbol{A}\left(\sum_{i\in\mathbf{I}} (Z_i \cap \{V(p) \geq 0\}) \right) \subset \boldsymbol{Z}_{\mathcal{F}},\ \boldsymbol{L}_{\mathcal{F}} := \boldsymbol{A}_{\mathcal{F}} \cap -\boldsymbol{A}_{\mathcal{F}} \subset \boldsymbol{Z}_{\mathcal{F}}, \text{ and } \pi = \mathrm{proj}_{(\boldsymbol{L}_{\mathcal{F}})^\perp}.$$

We will use extensively the following properties[11] for all $(p, q, z) \in \mathbb{R}^L \times \mathbb{R}^J \times \mathbb{R}^J$,

$$\left.\begin{aligned} & q \cdot \pi z = \pi q \cdot \pi z = \pi q \cdot z,\ \ker\pi = \boldsymbol{L}_{\mathcal{F}} \subset \ker V(p), \\ & V(p)\pi z = V(p)z, \text{ hence } W(p,q)\pi z = W(p,\pi q)\pi z = W(p,\pi q)z, \\ & \boldsymbol{L}_{\mathcal{F}_\pi} \subset \boldsymbol{A}_{\mathcal{F}_\pi} \subset \boldsymbol{Z}_{\mathcal{F}_\pi} \subset \operatorname{Im}\pi. \end{aligned}\right\} \tag{4.1}$$

Given the financial structure $\mathcal{F}$ and given $p \in \mathbb{R}^L$, we denote

$$\mathcal{V}_{\mathcal{F}}(p) := \left\{ (V(p)z_1, \ldots, V(p)z_I) : (z_i)_i \in \prod_i Z_i,\ \sum_{i\in\mathbf{I}} z_i = 0 \right\}.$$

Theorem 1 is a direct consequence of the following theorem.

Theorem 4. *Let* $\mathcal{F} = (V, (Z_i)_i)$ *be a standard financial structure.*

(a) $\mathcal{F}_\pi$ *satisfies the following property:* $\boldsymbol{Q}_{\mathcal{F}_\pi}(p) \cap \boldsymbol{Z}_{\mathcal{F}_\pi} \subset \boldsymbol{Q}_{\mathcal{F}}(p) \cap \boldsymbol{Z}_{\mathcal{F}}$ *for all* $p \in \mathbb{R}^L$. *Moreover,* $\mathcal{F}_\pi$ *satisfies the Financial Survival Assumption* **FS** *if* $\mathcal{F}$ *satisfies also* **FS**.
(b) *If* $\mathcal{F}$ *is closed then* $\mathcal{F}_\pi$ *is standard, reduced, and* $\mathcal{V}_{\mathcal{F}}(p) = \mathcal{V}_{\mathcal{F}_\pi}(p)$ *for all* $p \in \mathbb{R}^L$.
(c) *If* $\mathcal{F}$ *is closed, then for every standard economy* $\mathcal{E}$, *for every equilibrium* $(\bar{p}, \bar{x}, \bar{q}, \bar{y})$ *of* $(\mathcal{E}, \mathcal{F}_\pi)$, *there exists* $z^* \in \prod_i Z_i$ *such that* $(\bar{p}, \bar{x}, \pi\bar{q}, z^*)$ *is an equilibrium of* $(\mathcal{E}, \mathcal{F})$.
(d) $\mathcal{F}$ *and* $\mathcal{F}_\pi$ *are equivalent if* $\mathcal{F}$ *is closed, and we additionally assume* **PPP** *[Positive Payoff Portfolio]:* $\forall p \in \mathbb{R}^L, \forall i \in \mathbf{I}, \exists \zeta_{i,p} \in \boldsymbol{A}Z_i$, $V(p)\zeta_{i,p} \gg 0$.

The proof of Theorem 4 is given in the next section.

4.2. Proof of Theorem 4

4.2.1. Proof of Part (a) of Theorem 4

- We will show successively that $(i)\boldsymbol{Q}_{\mathcal{F}_\pi}(p) \cap \boldsymbol{Z}_{\mathcal{F}_\pi} \subset \boldsymbol{Q}_{\mathcal{F}_\pi}(p) \cap \operatorname{Im}\pi \subset \boldsymbol{Q}_{\mathcal{F}}(p)$ and $(ii)\boldsymbol{Z}_{\mathcal{F}_\pi} \subset \boldsymbol{Z}_{\mathcal{F}}$.
 The first inclusion of (i) is a consequence of the fact that $\boldsymbol{Z}_{\mathcal{F}_\pi} \subset \operatorname{Im}\pi$. We prove the second inclusion of (i) by contradiction. Assume that there is some $q \in \boldsymbol{Q}_{\mathcal{F}_\pi}(p) \cap \operatorname{Im}\pi$ such that $q \notin \boldsymbol{Q}_{\mathcal{F}}(p)$. Then there exists $i \in \mathbf{I}$ and $\zeta_i \in \boldsymbol{A}Z_i$ such that $W(p,q)\zeta_i > 0$. But $\pi\zeta_i \in \pi(\boldsymbol{A}Z_i) \subset \boldsymbol{A}(\pi Z_i) = \boldsymbol{A}(\mathrm{cl}\pi Z_i)$ since there exists $\lambda^k \downarrow 0$, $(z_i^k)_k \subset Z_i$, $\pi\zeta_i = \pi(\lim_k \lambda^k z_i^k) = \lim_k \lambda^k \pi(z_i^k) \in \boldsymbol{A}(\pi Z_i)$. Moreover, $W(p,q)(\pi\zeta_i) = W(p,q)\zeta_i > 0$, from (4.1) and the fact that $q = \pi q$ (since $q \in \operatorname{Im}\pi$). We thus have an arbitrage opportunity $\pi\zeta_i$ in $\mathcal{F}_\pi$, which contradicts the fact that $q \in \boldsymbol{Q}_{\mathcal{F}_\pi}(p)$; this ends the proof of (i). We now show that $\boldsymbol{Z}_{\mathcal{F}_\pi} \subset \boldsymbol{Z}_{\mathcal{F}}$. Indeed, let $y \in \boldsymbol{Z}_{\mathcal{F}_\pi}$, then $y = \pi z$ for some $z \in \boldsymbol{Z}_{\mathcal{F}}$ and $y = \pi z = \pi z - z + z \in \ker\pi + \boldsymbol{Z}_{\mathcal{F}} \subset \boldsymbol{Z}_{\mathcal{F}}$ since $\ker\pi \subset \boldsymbol{L}_{\mathcal{F}} \subset \boldsymbol{Z}_{\mathcal{F}}$.
- If $\mathcal{F}$ satisfies **FS** then so does $\mathcal{F}_\pi$: Let $i \in \mathbf{I}, p \in \mathbb{R}^L$ such that $p(0) = 0$, and $q \in \mathrm{cl}[\boldsymbol{Q}_{\mathcal{F}_\pi}(p) \cap \boldsymbol{Z}_{\mathcal{F}_\pi}]$. By the above property, we have $q \in \mathrm{cl}[\boldsymbol{Q}_{\mathcal{F}}(p) \cap \boldsymbol{Z}_{\mathcal{F}}]$. Since $\mathcal{F}$ satisfies **FS**, there exists $z_i \in Z_i$ such that $q \cdot z_i < 0$. Therefore the portfolio $\pi z_i \in \pi Z_i \subset \mathrm{cl}\pi Z_i$ and $q \cdot \pi z_i = q \cdot z_i$ (because $q \in \boldsymbol{Z}_{\mathcal{F}_\pi} \subset \operatorname{Im}\pi$ and π is an orthogonal projection). Hence $q \cdot \pi z_i < 0$.

4.2.2. Proof of Part (b) of Theorem 4

We will need the following two lemmas, the proofs of which are given in Appendix A.

[11] The first equality comes from the fact that $\pi q \cdot \pi z = \pi q \cdot z$, since $\pi q \in \operatorname{Im}\pi$ and $z - \pi z \in \ker\pi = (\operatorname{Im}\pi)^\perp$ since π is an orthogonal projection mapping; then by symmetry $q \cdot \pi z = \pi q \cdot \pi z = \pi q \cdot z$. The inclusion holds since $\boldsymbol{L}_{\mathcal{F}} := \boldsymbol{A}_{\mathcal{F}} \cap -\boldsymbol{A}_{\mathcal{F}} \subset \{V(p) \geq 0\} \cap -\{V(p) \geq 0\} = \ker V(p)$. The second set of equalities holds since $z - \pi z \in \ker\pi = \boldsymbol{L}_{\mathcal{F}} \subset \ker V(p)$.

Lemma 1. *Let $\mathcal{F}=(V,(Z_i)_i)$ be standard and closed. (a) We have*

$$\sum_{i\in\mathbf{I}}(\mathrm{cl}\pi Z_i\cap\{V(p)\geq v_i\})\subset\sum_{i\in\mathbf{I}}(Z_i\cap\{V(p)\geq v_i\})\quad \textit{for all } p\in\mathbb{R}^L,$$

for all $(v_i)_{i\in I}\in(\mathbb{R}^S)^I$.

(b) We have $\boldsymbol{A}_{\mathcal{F}_\pi}(p)\subset\boldsymbol{A}_{\mathcal{F}}$ *and* $\boldsymbol{L}_{\mathcal{F}_\pi}(p):=\boldsymbol{A}_{\mathcal{F}_\pi}(p)\cap-\boldsymbol{A}_{\mathcal{F}_\pi}(p)\subset\boldsymbol{L}_{\mathcal{F}}$ *for all* $p\in\mathbb{R}^L$.

Lemma 2. *Let $\mathcal{F}=(V,(Z_i)_i)$ be standard such that $\boldsymbol{L}_{\mathcal{F}}=\{0\}$, then $\mathcal{F}$ is reduced.*

We now give the proof of Part (*b*).

-]$\mathcal{F}_\pi$ is standard: $\mathcal{F}_\pi$ obviously satisfies **F1** and it remains to show that $\boldsymbol{A}_{\mathcal{F}_\pi}(p)$ does not depend on p. We show that for all $p\in\mathbb{R}^L$, $\pi(\boldsymbol{A}_{\mathcal{F}}(p))=\boldsymbol{A}_{\mathcal{F}_\pi}(p)$ and the desired result follows from the fact that $\boldsymbol{A}_{\mathcal{F}}(p)$ is independent of p (by **F0**). We first claim that

$$\pi(\boldsymbol{A}_{\mathcal{F}}(p))\subset\boldsymbol{A}\pi\sum_{i\in\mathbf{I}}(Z_i\cap\{V(p)\geq 0\})\subset\boldsymbol{A}\sum_{i\in\mathbf{I}}(\pi Z_i\cap\{V(p\geq 0)\})$$
$$\subset\boldsymbol{A}_{\mathcal{F}_\pi}(p).$$

Indeed, the first inclusion follows from the fact that if $\zeta\in\boldsymbol{A}_{\mathcal{F}}(p)$ then $\zeta=\lim_k\lambda^k z^k$ for some $\lambda^k\downarrow 0,(z^k)_k\subset\sum_{i\in\mathbf{I}}(Z_i\cap\{V(p)\geq 0\})$; hence $\pi\zeta=\pi(\lim_k\lambda^k z^k)=\lim_k\lambda^k\pi(z^k)\in\boldsymbol{A}\pi\sum_{i\in\mathbf{I}}(Z_i\cap\{V(p)\geq 0\})$. The second inclusion comes from (4.1). Finally, the last inclusion is immediate.

We now prove the converse inclusion $\boldsymbol{A}_{\mathcal{F}_\pi}(p)\subset\pi(\boldsymbol{A}_{\mathcal{F}}(p))$. From Lemma 1, taking $v_i=0$ $(i\in\mathbf{I})$, and then the asymptotic cones of both sides of the inclusion we get $\boldsymbol{A}_{\mathcal{F}_\pi}(p)\subset\boldsymbol{A}_{\mathcal{F}}(p)$. Thus $\pi(\boldsymbol{A}_{\mathcal{F}_\pi}(p))\subset\pi(\boldsymbol{A}_{\mathcal{F}}(p))$. Noticing that $\boldsymbol{A}_{\mathcal{F}_\pi}(p)\subset\mathrm{Im}\,\pi$, we conclude that $\boldsymbol{A}_{\mathcal{F}_\pi}(p)\subset\pi(\boldsymbol{A}_{\mathcal{F}}(p))$.
-]$\mathcal{F}_\pi$ is reduced: In view of Lemma 2, it suffices to show that $\boldsymbol{L}_{\mathcal{F}_\pi}=\{0\}$. We first have $\boldsymbol{L}_{\mathcal{F}_\pi}\subset\boldsymbol{L}_{\mathcal{F}}\cap\mathrm{Im}\,\pi$; indeed, $\boldsymbol{L}_{\mathcal{F}_\pi}\subset\boldsymbol{L}_{\mathcal{F}}$, by Lemma 1, and $\boldsymbol{L}_{\mathcal{F}_\pi}\subset\mathrm{Im}\,\pi$ by (4.1). Consequently, $\boldsymbol{L}_{\mathcal{F}_\pi}\subset\boldsymbol{L}_{\mathcal{F}}\cap\mathrm{Im}\,\pi=ker\pi\cap\mathrm{Im}\,\pi=\{0\}$, and the equality holds since $0\in\boldsymbol{L}_{\mathcal{F}_\pi}$.
-]$V_{\mathcal{F}}(p)\subset V_{\mathcal{F}_\pi}(p)$ for all $p\in\mathbb{R}^L$: Let $z_i\in Z_i$ $(i\in\mathbf{I})$ such that $\sum_{i\in\mathbf{I}}z_i=0$, let $y_i=\pi z_i\in\mathrm{cl}\pi Z_i$ $(i\in\mathbf{I})$ then $\sum_{i\in\mathbf{I}}y_i=\sum_{i\in\mathbf{I}}\pi z_i=\pi(\sum_{i\in\mathbf{I}}z_i)=0$, and $V(p)z_i=V(p)y_i$ for all i by (4.1).
-]$V_{\mathcal{F}_\pi}(p)\subset V_{\mathcal{F}}(p)$: Let $y:=(y_i)_i\in\prod_i\mathrm{cl}\pi Z_i$ such that $\sum_{i\in\mathbf{I}}y_i=0$. Then, by Lemma 1,

$$0=\sum_{i\in\mathbf{I}}y_i\in\sum_{i\in\mathbf{I}}(\mathrm{cl}\pi Z_i\cap\{V(p)\geq V(p)y_i\})\subset\sum_{i\in\mathbf{I}}(Z_i\cap\{V(p)\geq V(p)y_i\}).$$

Hence $0=\sum_{i\in\mathbf{I}}z_i$ for some $z_i\in Z_i$ $(i\in\mathbf{I})$ such that $V(p)z_i\geq V(p)y_i$ for every i. Noticing that $\sum_{i\in\mathbf{I}}z_i=\sum_{i\in\mathbf{I}}y_i(=0)$, we conclude that $V(p)z_i=V(p)y_i$ for every i.

4.2.3. Proof of Part (c) of Theorem 4

We first state and prove a claim.

Claim 4.1. *Under the assumption in Part (c), let $(q,y)\in\mathbb{R}^J\times(\prod_i\mathrm{cl}\pi Z_i)$ be arbitrage-free at p in $\mathcal{F}_\pi$ (i.e., for every $i\in\mathbf{I}$, there is no $\tilde{y}_i\in\mathrm{cl}\pi Z_i$ such that $W(p,q)\tilde{y}_i>W(p,q)y_i$), and $\sum_{i\in\mathbf{I}}y_i=0$.*

Then there exists $z_i^*\in Z_i$ $(i\in\mathbf{I})$ such that $\sum_{i\in\mathbf{I}}z_i^*=0$, and $W(p,q)y_i=W(p,\pi q)z_i^*$ for all $i\in\mathbf{I}$.

Proof. Let $(q,(y_i)_i)$ be arbitrage-free at p in $\mathcal{F}_\pi$, $\sum_{i\in\mathbf{I}}y_i=0$. Then, $(V(p)y_1,\ldots,V(p)y_I)\in\mathcal{V}_{\mathcal{F}_\pi}(p)$ and, by Part (*b*) of Theorem 4, $\mathcal{V}_{\mathcal{F}_\pi}(p)=\mathcal{V}_{\mathcal{F}}(p)$. Hence, $V(p)y_i=V(p)z_i^*$ for some $z_i^*\in Z_i$ $(i\in\mathbf{I})$ such that $\sum_{i\in\mathbf{I}}z_i^*=0$.

We end the proof by showing that $-q\cdot y_i=-\pi q\cdot z_i^*$ for all i. Since $\sum_{i\in\mathbf{I}}-q\cdot y_i=0=\sum_{i\in\mathbf{I}}-\pi q\cdot z_i^*$, it suffices to show that $-\pi q\cdot z_i^*\leq-q\cdot y_i$ for all $i\in\mathbf{I}$. If it is not true, there exists $i\in\mathbf{I}$, $-\pi q\cdot z_i^*>-q\cdot y_i$. Recalling that $V(p)z_i^*=V(p)y_i$ and using (4.1), we get $W(p,q)\pi z_i^*=W(p,\pi q)z_i^*>W(p,q)y_i$, which contradicts that $(q,(y_i)_i)$ is arbitrage-free in $\mathcal{F}_\pi$. □

We now give the proof of Part (*c*). Let $(\bar{p},\bar{x},\bar{q},\bar{y})$ be an equilibrium of $(\mathcal{E},\mathcal{F}_\pi)$, then $(\bar{q},\bar{y})$ is arbitrage-free at $\bar{p}$ in $\mathcal{F}_\pi$, under Local Non Satiation (**LNS**) (see Angeloni and Cornet, 2006). From the above Claim 4.1, there exists $z_i^*\in Z_i$ $(i\in\mathbf{I})$ such that $W(\bar{p},\pi\bar{q})z_i^*=W(\bar{p},\bar{q})\bar{y}_i$ for all i, $\sum_{i\in\mathbf{I}}z_i^*=0$. We show that $(\bar{p},\bar{x},\pi\bar{q},z^*)$ is an equilibrium of $(\mathcal{E},\mathcal{F})$. First, we have $(\bar{x}_i,z_i^*)\in B_i(\bar{p},\pi\bar{q},\mathcal{F})$ since $(\bar{x}_i,\bar{y}_i)\in B_i(\bar{p},\bar{q},\mathcal{F}_\pi)$ and $W(\bar{p},\bar{q})\bar{y}_i=W(\bar{p},\pi\bar{q})z_i^*$ for each $i\in\mathbf{I}$.

We complete the proof by showing that $B_i(\bar{p},\pi\bar{q},\mathcal{F})\cap(P_i(\bar{x})\times Z_i)=\varnothing$ for all $i\in\mathbf{I}$. Suppose it is not true, then there exist $i\in\mathbf{I}$ and $(x_i,z_i)\in B_i(\bar{p},\pi\bar{q},\mathcal{F})\cap(P_i(\bar{x})\times Z_i)$. Consequently, $(x_i,\pi z_i)\in B_i(\bar{p},\bar{q},\mathcal{F}_\pi)\cap(P_i(\bar{x})\times\mathrm{cl}\pi Z_i)$ since $W(\bar{p},\bar{q})\pi z_i=W(\bar{p},\pi\bar{q})z_i$ by (4.1). This contradicts the fact that $(\bar{p},\bar{x},\bar{q},\bar{y})$ is an equilibrium of $(\mathcal{E},\mathcal{F}_\pi)$.

4.2.4. Proof of Part (d) of Theorem 4

We will need the following two lemmas whose proofs are given in Appendix A.

Lemma 3. *Let $\mathcal{F}=(V,(Z_i)_i)$ be a standard financial structure satisfying* **PPP**. *If $(\bar{q},\bar{z})$ is arbitrage-free at $\bar{p}$ in $\mathcal{F}$ then $\bar{q}\in\boldsymbol{L}_{\mathcal{F}}^{\perp}=\mathrm{Im}\,\pi$.*

Lemma 4. *Assume that $e_i\in\mathrm{int}\,X_i$ and $\bar{p}(s)\neq 0$ for all $s\in\bar{\mathbf{S}}$, then*

$$B_i(\bar{p},\bar{q},\mathcal{F}_\pi)=\mathrm{cl}\{(x_i,y_i)\in X_i\times\pi Z_i:\bar{p}\square(x_i-e_i)\ll W(\bar{p},\bar{q})y_i\}.$$

We now give the proof of Part (*d*). In view of Part (*c*), it suffices to show that, for every standard exchange economy $\mathcal{E}$, if $(\bar{p},\bar{x},\bar{q},\bar{z})$ is an equilibrium of $(\mathcal{E},\mathcal{F})$, then $(\bar{p},\bar{x},\pi\bar{q},\pi\bar{z})$ is an equilibrium of $(\mathcal{E},\mathcal{F}_\pi)$. First, we have $\sum_{i\in\mathbf{I}}\pi\bar{z}_i=0$ as a direct consequence of $\sum_{i\in\mathbf{I}}\bar{z}_i=0$, the asset market clearing condition in $(\mathcal{E},\mathcal{F})$.

Second, we show that $(\bar{x}_i,\pi\bar{z}_i)\in B_i(\bar{p},\pi\bar{q},\mathcal{F}_\pi)$ for all $i\in\mathbf{I}$. Since $(\bar{x}_i,\bar{z}_i)\in B_i(\bar{p},\bar{q},\mathcal{F})$, it suffices to show that $W(\bar{p},\bar{q})\bar{z}_i=W(\bar{p},\pi\bar{q})\pi\bar{z}_i$. But, $W(\bar{p},\pi\bar{q})\pi\bar{z}_i=W(\bar{p},\pi\bar{q})\bar{z}_i$ (by (4.1)) and we end the proof by showing that $\pi\bar{q}=\bar{q}$. Indeed, $(\bar{q},\bar{z})$ is arbitrage-free at $\bar{p}$ in $\mathcal{F}$ since $(\bar{p},\bar{x},\bar{q},\bar{z})$ is an equilibrium of $(\mathcal{E},\mathcal{F})$ (see Angeloni and Cornet, 2006); thus, by Lemma 3, $\pi\bar{q}=\bar{q}$.

We now show that for each $i\in\mathbf{I}$, $(\bar{x}_i,\pi\bar{z}_i)$ solves agent i's problem in $(\mathcal{E},\mathcal{F}_\pi)$. Suppose on the contrary that there exist i and $(x_i,z_i)\in B_i(\bar{p},\pi\bar{q},\mathcal{F}_\pi)$, $x_i\in P_i(\bar{x})$. Recall that by **LNS** one has $\bar{p}(s)\neq 0$ for all $s\in\bar{\mathbf{S}}$. From Lemma 4, $(x_i,z_i)=\lim_n(x_i^n,\pi z_i^n)$ for some sequences $(x_i^n,z_i^n)_n\subset X_i\times Z_i$ such that

$$\bar{p}\square(x_i^n-e_i)-W(\bar{p},\pi\bar{q})(\pi z_i^n)\ll 0.$$

We have $W(\bar{p},\pi\bar{q})(\pi z_i^n)=W(\bar{p},\pi\bar{q})(z_i^n)=W(\bar{p},\bar{q})(z_i^n)$ (from (4.1) and the fact that $\pi\bar{q}=\bar{q}$, proved above). Consequently, $\bar{p}\square(x_i^n-e_i)-W(\bar{p},\bar{q})z_i^n=\bar{p}\square(x_i^n-e_i)-W(\bar{p},\pi\bar{q})(\pi z_i^n)\ll 0$, thus $(x_i^n,z_i^n)\in B_i(\bar{p},\bar{q},\mathcal{F})$. Recalling that $x_i\in P_i(\bar{x})$, $x_i=\lim_n x_i^n$ and using the fact that $P_i(\bar{x})$ is open (by Assumption **C**), we deduce that for n large enough $x_i^n\in P_i(\bar{x})$. The two assertions $(x_i^n,z_i^n)\in B_i(\bar{p},\bar{q},\mathcal{F})$ and $x_i^n\in P_i(\bar{x})$ contradict the fact that $(\bar{x}_i,\bar{z}_i)$ solves agent i's problem in $(\mathcal{E},\mathcal{F})$.

Appendix A.

A.1. Reducibility, WNMA, and Closedness

We recall, for $p\in\mathbb{R}^L$, the definition of $\mathcal{G}_{\mathcal{F}}(p)$ and we introduce the related set $\mathcal{G}'_{\mathcal{F}}(p)$.

$$\mathcal{G}_{\mathcal{F}}(p):=\left\{\left(v_1,\dots v_I,\sum_{i\in\mathbf{I}}z_i\right)\in\mathbb{R}^S\times\cdots\mathbb{R}^S\times\mathbb{R}^J : \forall i\in\mathbf{I}, z_i\in Z_i, V(p)z_i=v_i\right\},$$

$$\mathcal{G}'_{\mathcal{F}}(p):=\left\{\left(v_1,\dots v_I,\sum_{i\in\mathbf{I}}z_i\right)\in\mathbb{R}^S\times\cdots\mathbb{R}^S\times\mathbb{R}^J : \forall i\in\mathbf{I}, z_i\in Z_i, V(p)z_i\geq v_i\right\}.$$

The following result gives the relationship between the Reducibility, WNMA, and Closedness Conditions. We refer to Aouani et al. (2011) for counterexamples showing that the converse of Assertion (b) may not hold, and Assertion (d) may not hold if we remove the Positive Payoff Portfolio Assumption.

Proposition A.1. *Let $\mathcal{F}=(V,(Z_i)_i)$ be a financial structure and let $p\in\mathbb{R}^L$ be given.*

(a) *If $\mathcal{F}$ is reduced then it satisfies Hart's Weak No Market Arbitrage* **WNMA**, *i.e.*, **F2(ii)**.[12]
(b) *If $\mathcal{F}$ satisfies* **F2** *then it is closed, i.e., the set $\mathcal{G}_{\mathcal{F}}(p)$ is closed for all p.*
(c) *The set $\mathcal{G}_{\mathcal{F}}(p)$ is closed if and only if the set $\mathcal{G}'_{\mathcal{F}}(p)$ is closed.*
(d) *The set $\mathcal{G}'_{\mathcal{F}}(p)$ is closed if the two following assumptions hold:*
[space]$\forall i\in\mathbf{I}$, $\exists\zeta_{i,p}\in\boldsymbol{A}Z_i$, $V(p)\zeta_{i,p}\gg 0$ [*Positive Payoff Portfolio*].
[space]$\forall v=(v_i)_i\in\mathbb{R}^{SI}$, *the set* $\sum_{i\in\mathbf{I}}(Z_i\cap\{V(p)\geq v_i\})$ *is closed.*

Proof. *Part (a).* Let p be given. Let $\zeta_i\in\boldsymbol{A}Z_i\cap \ker V(p)$ $(i\in\mathbf{I})$ be such that $\sum_{i\in\mathbf{I}}\zeta_i=0$. Then $(\zeta_1,\dots,\zeta_I)\in\boldsymbol{A}(\mathcal{A}_{\mathcal{F}}(p,v))$. Since $\mathcal{F}$ is reduced, the set $\mathcal{A}_{\mathcal{F}}(p,v)$ is bounded, hence $\boldsymbol{A}(\mathcal{A}_{\mathcal{F}}(p,v))=\{0\}$ (see Rockafellar, 1970). Consequently, $\zeta_i=0\in-\boldsymbol{A}Z_i$ for all i.

Part (b). **F2(i)** $\Rightarrow$ Closedness: We first notice that $\mathcal{G}_{\mathcal{F}}(p)=f(\prod_i Z_i)$, where $f:\mathbb{R}^{JI}\to\mathbb{R}^{SI}\times\mathbb{R}^J$ is the linear mapping defined by $f(z_1,\dots,z_I)=(V(p)z_1,\dots,V(p)z_I,\sum_{i\in\mathbf{I}}z_i)$. ¿From Assumption **F2 (i)**, for all $i\in\mathbf{I}$, $Z_i=P_i+K_i$ where P_i is polyhedral convex and K_i is compact and convex. Hence, $\mathcal{G}_{\mathcal{F}}(p)=f(\prod_i Z_i)=f(\prod_i P_i)+f(\prod_i K_i)$ is closed since $f(\prod_i P_i)$ is closed (since it is polyhedral convex by Rockafellar (1970): Theorem 19.3, p. 174) and $f(\prod_i K_i)$ is compact.

-]**F2(ii)** $\Rightarrow$ Closedness: We can write the set $\mathcal{G}_{\mathcal{F}}(p)$ as follows

$$\mathcal{G}_{\mathcal{F}}(p)=\left\{\left(V(p)z_1,\dots,V(p)z_I,\sum_{i\in\mathbf{I}}z_i\right):\forall i, z_i\in Z_i\right\}=\sum_{i\in\mathbf{I}}X_i,$$
where $X_i=\left\{(0,\dots,0,V(p)z_i,0,\dots,0,z_i):z_i\in Z_i\right\}$.

To prove that the set $\mathcal{G}_{\mathcal{F}}(p)=\sum_{i\in\mathbf{I}}X_i$ is closed, it suffices to check that the sets $\boldsymbol{A}X_i$ $(i\in\mathbf{I})$ are weakly positively semi-independent (see Rockafellar, 1970:Theorem 9.1, p. 73). Indeed, let $w_i\in\boldsymbol{A}X_i=\{(0,\dots,0,V(p)\zeta_i,0,\dots,0,\zeta_i):\zeta_i\in\boldsymbol{A}Z_i\}$, such that $\sum_{i\in\mathbf{I}}w_i=0$; then there exist $\zeta_i\in\boldsymbol{A}Z_i$ $(i\in\mathbf{I})$ such that $V(p)\zeta_i=0$ for all i and $\sum_{i\in\mathbf{I}}\zeta_i=0$. Hence, $\zeta_i\in\boldsymbol{A}Z_i\cap \ker V(p)$ for each i and $\sum_{i\in\mathbf{I}}\zeta_i=0$. From **F2(ii)**, i.e., **WNMA**, we get $\zeta_i\in-\boldsymbol{A}Z_i$ for all i. Hence $w_i\in-\boldsymbol{A}X_i$ for all i.

Part (c). Assume that $\mathcal{G}'_{\mathcal{F}}(p)$ is closed and consider a sequence $(w^n)_n\subset\mathcal{G}_{\mathcal{F}}(p)$, converging to some w, i.e., $w^n=(V(p)z_1^n,\dots,V(p)z_I^n,\sum_{i\in\mathbf{I}}z_i^n)\to w=(v_1,\dots,v_I,z)$, with $z_i^n\in Z_i$ $(i\in\mathbf{I})$. Then $(w^n)_n\subset\mathcal{G}_{\mathcal{F}}(p)\subset\mathcal{G}'_{\mathcal{F}}(p)$, hence $w:=\lim_{n\to\infty}w_n\in\mathcal{G}'_{\mathcal{F}}(p)$ since $\mathcal{G}'_{\mathcal{F}}(p)$ is closed. Hence $z=\sum_{i\in\mathbf{I}}z_i$ for some $z_i\in Z_i$, $V(p)z_i\geq v_i$ for all $i\in\mathbf{I}$. But $\sum_{i\in\mathbf{I}}v_i=\sum_{i\in\mathbf{I}}\lim_n V(p)z_i^n=\lim_n V(p)(\sum_{i\in\mathbf{I}}z_i^n)=V(p)z=\sum_{i\in\mathbf{I}}V(p)z_i$. Consequently, $v_i=V(p)z_i$ for all $i\in I$ and $w=(V(p)z_1,\dots,V(p)z_I,\sum_{i\in\mathbf{I}}z_i)\in\mathcal{G}_{\mathcal{F}}(p)$.

Conversely, assume $\mathcal{G}_{\mathcal{F}}(p)$ closed and consider a sequence $(w'^n)_n\subset\mathcal{G}'_{\mathcal{F}}(p)$, converging to some w', i.e., $w'^n=(v_1'^n,\dots,v_I'^n,\sum_{i\in\mathbf{I}}z_i^n)\underset{n\to\infty}{\to}w'=(v'_1,\dots,v'_I,z)$, with $z_i^n\in Z_i$ and $V(p)z_i^n\geq v_i'^n$ for all $i\in\mathbf{I}$ and all $n\in\mathbb{N}$. For each $i\in\mathbf{I}$, the sequence $(V(p)z_i^n)_n$ is bounded below since $V(p)z_i^n\geq v_i'^n$ for every n and the converging sequence $(v_i'^n)_n$ is bounded; moreover the sequence $(V(p)z_i^n)_n$ is bounded since $(\sum_{i\in\mathbf{I}}V(p)z_i^n)_n$ converges (towards $V(p)z$). Without any loss of generality, we can thus assume that for all $i\in\mathbf{I}$, the sequence $(V(p)z_i^n)_n$ converges to some $v_i\in\mathbb{R}^S$ satisfying $v_i\geq v'_i$. Now we consider the sequence $w^n=(V(p)z_1^n,\dots,V(p)z_I^n,\sum_{i\in\mathbf{I}}z_i^n)$ in $\mathcal{G}_{\mathcal{F}}(p)$. From above, $w=(v_1,\dots,v_I,z)=\lim_{n\to\infty}w^n\in\mathcal{G}_{\mathcal{F}}(p)$ since $\mathcal{G}_{\mathcal{F}}(p)$ is closed. Hence, $v_i=V(p)z_i$ and $z=\sum_{i\in\mathbf{I}}z_i$ for some $z_i\in Z_i$ $(i\in\mathbf{I})$. Recall that $V(p)z_i=v_i\geq v'_i$ for each $i\in\mathbf{I}$ and that $w'=(v'_1,\dots,v'_I,z)=(v'_1,\dots,v'_I,\sum_{i\in\mathbf{I}}z_i)$, hence $w'\in\mathcal{G}'_{\mathcal{F}}(p)$.

Part (d). Let $(w^n)_n$ be a sequence in $\mathcal{G}'_{\mathcal{F}}(p)$ converging to some $w=(v_1,\dots,v_I,z)$, i.e., $w^n=(v_1^n,\dots,v_I^n,\sum_{i\in\mathbf{I}}z_i^n)$, with $z_i^n\in Z_i$ and $V(p)z_i^n\geq v_i^n$ for all $i\in\mathbf{I}$ and all $n\in\mathbb{N}$. By assumption, for every $i\in\mathbf{I}$, there exist a Positive Payoff Portfolio $\zeta_{i,p}\in\boldsymbol{A}Z_i$ such that $V(p)\zeta_{i,p}\gg 0$. We claim that

$$z_i^n+t_i^n\zeta_{i,p}\in Z_i\cap\{V(p)\geq v_i\},\quad \text{where } t_i^n=\frac{\max_s|(v_i^n-v_i)(s)|}{\min_s(V_s(p)\cdot\zeta_{i,p})}>0$$
$$\text{and}\quad t_i^n\underset{n\to\infty}{\to}0.$$

Indeed, $z_i^n+t_i^n\zeta_{i,p}\in Z_i+\boldsymbol{A}Z_i\subset Z_i$, and for every $s\in\mathbf{S}$, we have

$$V_s(p)\cdot(z_i^n+t_i^n\zeta_{i,p})=V_s(p)\cdot z_i^n+t_i^nV_s(p)\cdot\zeta_{i,p}\geq v_i^n(s)+|v_i^n(s)-v_i(s)|\geq v_i(s).$$

Hence, $z=\lim_{n\to\infty}\sum_{i\in\mathbf{I}}(z_i^n+t_i^n\zeta_{i,p})\in\sum_{i\in\mathbf{I}}(Z_i\cap\{V(p)\geq v_i\})$ (since by assumption the set is closed). Thus $z=\sum_{i\in\mathbf{I}}z_i$ for some $z_i\in Z_i$, $V(p)z_i\geq v_i$ $(i\in\mathbf{I})$, that is, $w=(v_1,\dots,v_I,z)\in\mathcal{G}'_{\mathcal{F}}(p)$. □

A.2. Proof of Lemma 1

We prepare the proof with two claims. We let

$$\mathcal{V}:=\{v=(v_i)_{i\in I}\in(\mathbb{R}^S)^I : Z_i\cap\{V(p)\geq v_i\}\neq\varnothing\quad\text{for all } i\in\mathbf{I}\}.$$

Claim A.1. *Let $\mathcal{F}=(V,(Z_i)_i)$ be standard and closed and let p be given. Then*

$$\boldsymbol{A}_{\mathcal{F}}(p,v):=\boldsymbol{A}\left(\sum_{i\in\mathbf{I}}(Z_i\cap\{V(p)\geq v_i\})\right)=\boldsymbol{A}_{\mathcal{F}}\quad\text{for all } v\in\mathcal{V}.$$

Proof. Notice first that $0\in\mathcal{V}$ and that $\boldsymbol{A}_{\mathcal{F}}:=\boldsymbol{A}_{\mathcal{F}}(p,0)$ by definition. Thus, it suffices to show that $\boldsymbol{A}\left(\sum_{i\in\mathbf{I}}(Z_i\cap\{V(p)\geq v_i\})\right)\subset\boldsymbol{A}\left(\sum_{i\in\mathbf{I}}(Z_i\cap\{V(p)\geq w_i\})\right)$ for all v,w in $\mathcal{V}$.

Let $\zeta\in\boldsymbol{A}\left(\sum_{i\in\mathbf{I}}(Z_i\cap\{V(p)\geq v_i\})\right)$, then $\zeta=\lim_{n\to\infty}\lambda^n\sum_{i\in\mathbf{I}}z_i^n$ for some $z_i^n\in Z_i\cap\{V(p)\geq v_i\}$, $\lambda^n>0$, and $\lambda^n\downarrow 0$. We need to show that

[12] The converse of the Assertion (a) may not be true. Consider the financial structure $\mathcal{F}$ defined by $V=\begin{bmatrix}0&0&1\end{bmatrix}$, $I=2$, and $Z_1=Z_2=\{(z_1,z_2,z_3)\in\mathbb{R}^3: z_1\in\mathbb{R},\ z_2\in\mathbb{R},\ z_3\geq z_1^2\}$; then $\mathcal{F}$ satisfies **F2(ii)** and is not reduced. Keeping the same payoff matrix and modifying the portfolio sets as follows $Z'_1=Z'_2=\{(z_1,z_2,z_3)\in\mathbb{R}^3: z_1\geq 0,\ z_2\in\mathbb{R},\ z_3\in\mathbb{R}\}$ allows to construct a financial structure $\mathcal{F}'$, which satisfies **F2(i)** and is not reduced.

$\zeta\in\boldsymbol{A}\left(\sum_{i\in\mathbf{I}}(Z_i\cap\{V(p)\geq w_i\})\right)$, that is,

$$\zeta+\sum_{i\in\mathbf{I}}z_i\in\sum_{i\in\mathbf{I}}(Z_i\cap\{V(p)\geq w_i\})\quad\text{for all } z_i\in Z_i,$$

such that $V(p)z_i\geq w_i\,(i\in\mathbf{I})$.

From above, $\zeta+\sum_{i\in\mathbf{I}}z_i=\lim_{n\to\infty}\sum_{i\in\mathbf{I}}(\lambda_n z_i^n+(1-\lambda_n)z_i)$. Notice that, $y_i^n:=\lambda^n z_i^n+(1-\lambda^n)z_i\in Z_i$ for n large enough, so that $\lambda^n\in[0,1]$ (because z_i^n and z_i are in Z_i, and Z_i is convex). Furthermore $V(p)y_i^n\geq\lambda^n v_i+(1-\lambda^n)w_i$. Consequently

$$\left(\lambda^n v_1+(1-\lambda^n)w_1,\ldots,\lambda^n v_I+(1-\lambda^n)w_I,\sum_{i\in\mathbf{I}}y_i^n\right)\in\mathcal{G}'_{\mathcal{F}}(p),$$

Since $\mathcal{F}$ is closed, the set $\mathcal{G}_{\mathcal{F}}(p)$ is closed and the set $\mathcal{G}'_{\mathcal{F}}(p)$ is also closed by Proposition A.1. Thus $(w_1,\ldots,w_I,\zeta+\sum_{i\in\mathbf{I}}z_i)\in\mathcal{G}'_{\mathcal{F}}(p)$, i.e., $\zeta+\sum_{i\in\mathbf{I}}z_i\in\sum_{i\in\mathbf{I}}(Z_i\cap\{V(p)\geq w_i\})$. □

Claim A.2. Let $\mathcal{F}=(V,(Z_i)_i)$ be standard and closed, one has

$$\sum_{i\in\mathbf{I}}(\pi Z_i\cap\{V(p)\geq v_i\})\subset\sum_{i\in\mathbf{I}}(Z_i\cap\{V(p)\geq v_i\})\quad\text{for all } p\in\mathbb{R}^L,$$

for all $(v_i)_{i\in I}\in(\mathbb{R}^S)^I$.

Proof. Assume that $\sum_{i\in\mathbf{I}}(\pi Z_i\cap\{V(p)\geq v_i\})\neq\varnothing$, otherwise the proof is immediate. We show successively that

$$\sum_{i\in\mathbf{I}}(\pi Z_i\cap\{V(p)\geq v_i\})\subset\sum_{i\in\mathbf{I}}\pi(Z_i\cap\{V(p)\geq v_i\})\tag{A.1}$$

$$\subset ker\pi+\sum_{i\in\mathbf{I}}(Z_i\cap\{V(p)\geq v_i\})\tag{A.2}$$

$$\subset\sum_{i\in\mathbf{I}}(Z_i\cap\{V(p)\geq v_i\}).\tag{A.3}$$

To prove (A.1), it suffices to notice that $\pi Z_i\cap\{V(p)\geq v_i\}\subset\pi(Z_i\cap\{V(p)\geq v_i\})$ for all $i\in\mathbf{I}$; indeed, let $y_i\in\pi Z_i\cap\{V(p)\geq v_i\}$, then $y_i=\pi z_i$ for some $z_i\in Z_i$, and $V(p)y_i\geq v_i$. But $V(p)z_i=V(p)(\pi z_i)=V(p)y_i$, by (4.1). Thus $z_i\in Z_i\cap\{V(p)\geq v_i\}$ and $y_i=\pi z_i\in\pi(Z_i\cap\{V(p)\geq v_i\})$.

To prove (A.2), let $y=\sum_{i\in\mathbf{I}}\pi z_i$ with $z_i\in Z_i\cap\{V(p)\geq v_i\}$. Then $y=\pi z=(\pi z-z)+z$ with $\pi z-z\in ker\pi$ and $z=\sum_{i\in\mathbf{I}}z_i\in\sum_{i\in\mathbf{I}}(Z_i\cap\{V(p)\geq v_i\})$. This ends the proof of (A.2).

The last inclusion (A.3) comes from the fact that

$$ker\pi=\boldsymbol{L}_{\mathcal{F}}\subset\boldsymbol{A}_{\mathcal{F}}=\boldsymbol{A}\left(\sum_{i\in\mathbf{I}}(Z_i\cap\{V(p)\geq v_i\})\right),$$

where the first equality holds by definition of π, the inclusion is immediate and the second equality holds by Claim A.1 since $\sum_{i\in\mathbf{I}}(Z_i\cap\{V(p)\geq v_i\})\neq\varnothing$ (by (A.1), (A.2) and the assumption that $\sum_{i\in\mathbf{I}}(\pi Z_i\cap\{V(p)\geq v_i\})\neq\varnothing$). Consequently,

$$ker\pi+\sum_{i\in\mathbf{I}}(Z_i\cap\{V(p)\geq v_i\})\subset\boldsymbol{A}\left(\sum_{i\in\mathbf{I}}(Z_i\cap\{V(p)\geq v_i\})\right)+\sum_{i\in\mathbf{I}}(Z_i\cap\{V(p)\geq v_i\})$$
$$\subset\mathrm{cl}\sum_{i\in\mathbf{I}}(Z_i\cap\{V(p)\geq v_i\})=\sum_{i\in\mathbf{I}}(Z_i\cap\{V(p)\geq v_i\}).$$

The last equality follows from the closedness assumption. This ends the proof of the claim. □

Proof of Lemma 1. *Part (a).* If the left-hand side of the inclusion is empty then the result is trivial. Otherwise, let $y_i\in(\mathrm{cl}\pi Z_i)\cap\{V(p)\geq v_i\}$ $(i\in\mathbf{I})$. Take $v_i^n\uparrow v_i$ such that $v_i\gg v_i^n$ for every n. Pick $\bar y_i\in\mathrm{ri}\,\pi Z_i$ and consider $y_i^n=(1-\lambda^n)y_i+\lambda^n\bar y_i$ with $0<\lambda^n<\frac{1}{n}$ small enough so that $V(p)y_i^n\gg v_i^n$. Then $y_i^n\in[\bar y_i,y_i)\subset\mathrm{ri}\,\pi Z_i$ since $y_i\in\mathrm{cl}\pi Z_i$ and $\bar y_i\in\mathrm{ri}\,\pi Z_i$ (Theorem 6.1, p. 45 in Rockafellar, 1970). Thus $y_i^n\in\pi Z_i\cap\{V(p)\geq v_i^n\}$ and, by Claim A.2,

$$\sum_{i\in\mathbf{I}}y_i^n\in\sum_{i\in\mathbf{I}}(Z_i\cap\{V(p)\geq v_i^n\})\quad\text{hence}\quad\left(v_1^n,\ldots,v_I^n,\sum_{i\in\mathbf{I}}y_i^n\right)\in\mathcal{G}'_{\mathcal{F}}(p).$$

Since $\mathcal{F}$ is closed, the set $\mathcal{G}_{\mathcal{F}}(p)$ is closed and the set $\mathcal{G}'_{\mathcal{F}}(p)$ is also closed by Proposition A.1. Thus

$$\left(v_1,\ldots,v_I,\sum_{i\in\mathbf{I}}y_i\right)=\lim_n\left(v_1^n,\ldots,v_I^n,\sum_{i\in\mathbf{I}}y_i^n\right)\in\mathrm{cl}\mathcal{G}'_{\mathcal{F}}(p)=\mathcal{G}'_{\mathcal{F}}(p),$$

and $\sum_{i\in\mathbf{I}}y_i\in\sum_{i\in\mathbf{I}}(Z_i\cap\{V(p)\geq v_i\})$.

Part (b). Taking $v_i=0\,(i\in\mathbf{I})$, from Part (a) one gets, for all $p\in\mathbb{R}^L$,

$$\sum_{i\in\mathbf{I}}((\mathrm{cl}\pi Z_i)\cap\{V(p)\geq 0\})\subset\sum_{i\in\mathbf{I}}(Z_i\cap\{V(p)\geq 0\}),\quad\text{hence}$$

$$\boldsymbol{A}_{\mathcal{F}_\pi}(p):=\boldsymbol{A}\left(\sum_{i\in\mathbf{I}}((\mathrm{cl}\pi Z_i)\cap\{V(p)\geq 0\})\right)$$
$$\subset\boldsymbol{A}\left(\sum_{i\in\mathbf{I}}(Z_i\cap\{V(p)\geq 0\})\right):=\boldsymbol{A}_{\mathcal{F}},$$

and clearly $\boldsymbol{L}_{\mathcal{F}_\pi}(p):=\boldsymbol{A}_{\mathcal{F}_\pi}(p)\cap-\boldsymbol{A}_{\mathcal{F}_\pi}(p)\subset\boldsymbol{A}_{\mathcal{F}}\cap-\boldsymbol{A}_{\mathcal{F}}:=\boldsymbol{L}_{\mathcal{F}}$. □

A.3. Proof of Lemma 2

To show that $\mathcal{F}$ is reduced, that is, the set $\mathcal{A}_{\mathcal{F}}(p,v)$ is bounded, it suffices to show that $\boldsymbol{A}(\mathcal{A}_{\mathcal{F}}(p,v))=\{0\}$ (see Rockafellar, 1970). We have

$$\boldsymbol{A}(\mathcal{A}_{\mathcal{F}}(p,v))=\left\{(\zeta_1,\ldots,\zeta_I)\in\prod_{i\in I}\boldsymbol{A}Z_i:\forall i,\,V(p)\zeta_i\geq 0,\,\sum_{i\in I}\zeta_i=0\right\}.$$

Let $\zeta=(\zeta_1,\ldots,\zeta_I)\in\boldsymbol{A}(\mathcal{A}_{\mathcal{F}}(p,v))$. Then $\zeta=0$ since, for all i:

$$\zeta_i=-\sum_{k\neq i}\zeta_k\in(\boldsymbol{A}Z_i\cap\{V(p)\geq 0\})\cap-\left(\sum_{k\neq i}(\boldsymbol{A}Z_k\cap\{V(p)\geq 0\})\right)$$
$$\subset\boldsymbol{A}_{\mathcal{F}}\cap-\boldsymbol{A}_{\mathcal{F}}=\{0\}.$$

A.4. Proof of Lemma 3

We show that[13] $\bar q\in-(\boldsymbol{A}_{\mathcal{F}})^o$ and we then deduce that $-(\boldsymbol{A}_{\mathcal{F}})^o\subset(\boldsymbol{L}_{\mathcal{F}})^\perp=\mathrm{Im}\,\pi$ since $\boldsymbol{L}_{\mathcal{F}}\subset\boldsymbol{A}_{\mathcal{F}}$, $\boldsymbol{L}_{\mathcal{F}}=ker\pi$ (by definition) and $(ker\pi)^\perp=\mathrm{Im}\,\pi$. By contraposition; let $(\bar q,\bar z)$ be arbitrage-free at $\bar p$ in $\mathcal{F}$, $\bar q\notin-(\boldsymbol{A}\sum_{i\in\mathbf{I}}(Z_i\cap\{V(\bar p)\geq 0\}))^o$. Then there exists $\zeta\in\boldsymbol{A}\sum_{i\in\mathbf{I}}(Z_i\cap\{V(\bar p)\geq 0\})$ such that $-\bar q\cdot\zeta>0$. Thus, for every $n\in\mathbb{N}$, $n^2\zeta=\sum_{i\in\mathbf{I}}z_i^n$ for some $z_i^n\in Z_i\cap\{V(\bar p)\geq 0\}$. Therefore $-\bar q\cdot\sum_{i\in\mathbf{I}}(z_i^n/n)=-n\bar q\cdot\zeta\underset{n\to\infty}{\to}+\infty$. Hence, without any loss of generality, for some agent, say $i=1$, $-\bar q\cdot(z_1^n/n)\underset{n\to\infty}{\to}+\infty$. By **PPP**, there exists $\zeta_1\in\boldsymbol{A}Z_1$ such that $V(\bar p)\zeta_1\gg 0$. Define

$$y_1^n:=\frac{1}{n}z_1^n+(1-\frac{1}{n})(\bar z_1+\zeta_1).$$

[13] If $A\subset\mathbb{R}^J$, we denote $A^o:=\{q\in\mathbb{R}^J: q\cdot a\leq 0 \text{ for all } a\in A\}$ the negative polar of A.

We end the proof by showing that **(i)** $y_1^n \in Z_1$, and **(ii)** for n large enough, $W(\bar{p},\bar{q})y_1^n > W(\bar{p},\bar{q})\bar{z}_1$ and both assertions contradict the fact that $(\bar{q},\bar{z})$ is arbitrage-free at $\bar{p}$ in $\mathcal{F}$ (see Angeloni and Cornet, 2006). First, since $\zeta_1 \in \boldsymbol{A}Z_1$, one has $\bar{z}_1+\zeta_1 \in Z_1$, and since $z_1^n \in Z_1$ and $\bar{z}_1+\zeta_1 \in Z_1$, the convexity of Z_1 (since $\mathcal{F}$ is standard) allows to conclude that y_1^n belongs to Z_1. Second, since $-\bar{q}\cdot(z_1^n/n) \underset{n\to\infty}{\rightarrow} +\infty$, one has, for n large enough $-\bar{q}\cdot y_1^n = -\bar{q}\cdot \frac{1}{n}z_1^n + -\bar{q}\cdot(1-\frac{1}{n})(\bar{z}_1+\zeta_1) > -\bar{q}\cdot\bar{z}_1$.

Finally, since $z_1^n \in \{V(\bar{p}) \geq 0\}$ and $V(\bar{p})\zeta_1 \gg 0$, one has, for n large enough

$$V(\bar{p})y_1^n = V(\bar{p})\left(\frac{1}{n}z_1^n + (1-\frac{1}{n})(\bar{z}_1+\zeta_1)\right) \geq (1-\frac{1}{n})V(\bar{p})(\bar{z}_1+\zeta_1)$$

$$\gg V(\bar{p})\bar{z}_1.$$

Hence, for n large enough, $W(\bar{p},\bar{q})y_1^n > W(\bar{p},\bar{q})\bar{z}_1$. This ends the proof of the lemma.

A.5. *Proof of Lemma 4*

We first choose $\delta = (\delta(s))_{s\in\bar{\mathbf{S}}} \in \mathbb{R}^L$ such that **(i)** $e_i - \delta \in X_i$ and **(ii)** $p(s)\cdot\delta(s) > 0$ for every $s \in \bar{\mathbf{S}}$; indeed, take $\delta = \lambda p$ for $\lambda > 0$ small enough so that $e_i - \delta \in X_i$ (since $e_i \in \operatorname{int} X_i$) and $p(s)\cdot\delta(s) = \lambda p(s)\cdot p(s) > 0$, since $p(s) \neq 0$ for all $s \in \bar{\mathbf{S}}$.

Let $(x_i, y_i) \in B_i(p, q, \mathcal{F}_\pi)$. Let $\alpha \in (0,1)$. Then $x_i^\alpha := \alpha x_i + (1-\alpha)(e_i - \delta) \in X_i$ since $x_i \in X_i$, $e_i - \delta \in X_i$ and X_i is convex, and $\alpha y_i \in \mathrm{cl}\pi Z_i$ since $0 \in \mathrm{cl}\pi Z_i$, $y_i \in \mathrm{cl}\pi Z_i$, and $\mathrm{cl}\pi Z_i$ is convex. We claim that,

$$p\Box(x_i^\alpha - e_i) - W(p,q)(\alpha y_i) \ll 0.$$

Indeed, $p\Box(x_i^\alpha - e_i) - W(p,q)(\alpha y_i) = \alpha(p\Box(x_i - e_i) - W(p,q)y_i) - (1-\alpha)p\Box\delta$. Since $(x_i, y_i) \in B_i(p,q,\mathcal{F}_\pi)$, i.e., $p\Box(x_i - e_i) - W(p,q)y_i \leq 0$, and $\alpha > 0$, the first term is nonpositive. Since $p\Box\delta \gg 0$ (from above) and $\alpha < 1$, the second term satisfies $-(1-\alpha)p\Box\delta \ll 0$. This ends the proof of the claim.

Consequently, there exists $y_i^\alpha \in \pi Z_i$ such that $\|y_i^\alpha - y_i\| \leq (1-\alpha)\|y_i\|$ and

$$p\Box(x_i^\alpha - e_i) - W(p,q)y_i^\alpha \ll 0.$$

Noticing that, $(x_i^\alpha, y_i^\alpha) \rightarrow (x_i, y_i)$ when $\alpha \rightarrow 1$, we get the desired result.

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